SOLVED EXERCISE 2.1

1. Find the discriminant of the following given quadratic equations:

(i) $2x^2 + 3x - 1 = 0$

Solution:

$2x^2 + 3x - 1 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 2, b = 3, c = -1$

Disc. $= b^2 - 4ac$

$= (3)^2 - 4(2)(-1)$

$= 9 + 8$

$= 17$

(ii) $6x^2 - 8x + 3 = 0$

Solution:

$6x^2 - 8x + 3 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 6, b = -8, c = 3$

Disc. $= b^2 - 4ac$

$= (-8)^2 - 4(6)(3)$

$= 64 - 72$

$= -8$

(iii) $9x^2 - 30x + 25 = 0$

Solution:

$9x^2 - 30x + 25 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 9, b = -30, c = 25$

Disc. $= b^2 - 4ac$

$= (-30)^2 - 4(4)(25)$

$= 900 - 900$

$= 0$

(iv) $4x^2 - 7x - 2 = 0$

Solution:

$4x^2 - 7x - 2 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 4, b = -7, c = -2$

Disc. $= b^2 - 4ac$

$= (-7)^2 - 4(4)(-2)$

$= 49 + 32 = 81$

2. Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

(i) $x^2 + 23x + 120 = 0$

Solution:

$x^2 + 23x + 120 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 1, b = -23, c = 120$

Disc. $= b^2 - 4ac$

$= (-23)^2 - 4(1)(120)$

$= 529 - 480$

$= 49$

$= (7)^2 > 0$

As the disc. is positive and is a perfect square. Therefore the roots are rational (real) and unequal. Verification by solving the equation.

$x^2 - 23x + 120 = 0$

$x^2 - 15x - 8x + 120 = 0$

$x(x - 15) - 8(x - 15) = 0$

$(x - 8)(x - 15) = 0$

Either $x - 8 = 0$ or $x - 15 = 0$

$x = 8$ $\quad$ $x = 15$

thus, the roots are rational (real) and unequal.

(ii) $2x^2 + 3x + 7 = 0$

Solution:

$2x^2 + 3x + 7 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 2, b = 3, c = 7$

Disc. $= b^2 - 4ac$

$= (3)^2 - 4(2)(7)$

$= 9 - 56$

$= -47 < 0$

As the Disc. is negative.

Therefore the roots are imaginary and unequal.

Verification by solving the equation

$2x^2 + 3x + 7 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9-56}}{4}$$

$$= \frac{-3 \pm \sqrt{-47}}{4}$$

Thus, the roots are imaginary and unequal.

(iii) $16x^2 - 24x + 9 = 0$

Solution:

$16x^2 - 24x + 9 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 16, b = -24, c = 9$

Disc. $= b^2 - 4ac$

$= (-24)^2 - 4(16)(9)$

$= 576 - 576$

$= 0$

As the Disc. is zero.

Therefore the roots of the equation are real and equal.

Verification by solving the equation

$16x^2 - 24x + 9 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$= \frac{24 \pm \sqrt{576 - 576}}{32}$$

$$= \frac{24 \pm \sqrt{0}}{32}$$

$$= \frac{24}{32} = \frac{3}{4}$$

Thus, the roots are real and unequal.

(iv) $3x^2 + 7x - 13 = 0$

Solution:

$3x^2 + 7x - 13 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 3, b = 7, c = -13$

Disc. $= b^2 - 4ac$

$= (7)^2 - 4(3)(-13)$

$= 49 + 156$
$= 205 \quad > 0$

As the Disc. is positive and is not a perfect square.
Therefore the roots are irrational (real) and unequal.
Verification by solving the equation

$$3x^2 + 7x - 13 = 0$$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$= \frac{-7 \pm \sqrt{205}}{6}$$

Thus, the roots are irrational (real) and unequal.

3. For what value of A, the expression $k^2 x^2 + 2(k + 1)x + 4$ is perfect square.

Solution:

$$k^2 x^2 + 2(k + 1)x + 4 = 0$$

Compare it with

$$ax^2 + bx + c = 0$$

Here $a = k^2, b = 2(k + 1), c = 4$

Disc. $= b^2 - 4ac$
$= [2(k + 1)]^2 - 4(k^2)(4)$
$= 4(k + 1)^2 - 16k^2$
$= 4(k^2 + 2k + 1) - 16k^2$
$= 4k^2 + 8k + 4 - 16k^2$
$= -12k^2 + 8k + 4 = 0$

As the disc of the given expression is a perfect square. Therefore the roots are rational and equal.

So $\quad$ Disc $= 0$

$$-12K^2 + 8K + 4 = 0$$

$$-(12K^2 + 8K + 4) = 0$$

$$\Rightarrow \quad 12K^2 - 8K - 4 = 0$$

$$12K^2 - 12K + 4K - 4 = 0$$

$$12K(K - 1) + 4(K - 1) = 0$$

$$(12K + 4)(K - 1) = 0$$

Either $\quad 12K+4=0 \quad$ or $\quad K-1=0$

$12K = -4$ $\quad\quad\quad$ $K = 1$

$$K = -\frac{4}{12}$$

$$K = -\frac{1}{3}$$

4. Find the value of k, if the roots of the following equations are equal.

(i) $(2k+1)x^2 + 3kx\ 3 = 0$

Solution:

$$(2k+1)x^2 + 3Kx + 3 = 0$$

Here $a = 2k+1$, $b = 3k$, $c = 3$

As the roots are equal, So

$$\text{Disc.} = 0$$

$$b^2 - 4ac = 0$$

$$(3k)^2 - 4(2k+1)(3) = 0$$

$$9k^2 - 12(2k+1) = 0$$

$$9k^2 - 24k + 12 = 0$$

$$3(3k^2 - 8k + 4) = 0$$

$$\Rightarrow \quad 3k^2 - 8k + 4 = 0$$

$$3k^2 - 6k - 2k + 4 = 0$$

$$(k-2) - 2(k-2) = 0$$

$$(3k-2)(k-2) = 0$$

Either $3k-2=0$ or $k-2=0$

$3k=2$ $\quad\quad$ $k=2$

$$k = \frac{2}{3}$$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

Solution:

$$x^2 + 2(k+2)x + (3k+4) = 0$$

Here $a = 1$, $b = 2(k+2)$, $c = 3k+4$

As the roots are equal, So

$$\text{Disc.} = 0$$

$$b^2 - 4ac = 0$$

$$[2(k+2)]^2 - 4(1)(3k+4) = 0$$

$$4(k+2)^2 - 4(3k+4) = 0$$

$$4(k^2 + 4k + 4) - 12k - 16 = 0$$

$$4k^2 + 16k + 16 - 12k - 16 = 0$$

$$4k^2 + 4k = 0$$

$$4k(k+1) = 0$$

Either $4k = 0$ or $k+1 = 0$

$K = 0$ $K = -1$

(iii) $(3k + 2)\, x^2 - 5\,(k + 1)\, x + (2k + 3) = 0$

Solution:

$(3k + 2)\, x^2 - 5\,(k + 1)\, x + (2k + 3) = 0$

Here $a = 3k+2$, $b = -5(k+1)$, $c = (2k+3)$

As the roots are equal, So

$$\text{Disc.} = 0$$

$$b^2 - 4ac = 0$$

$$[-5(k+1)]^2 - 4(3k+2)(2k+3) = 0$$

$$25(k^2 + 2k + 1) - 4(6k^2 + 13k + 6) = 0$$

$$25k^2 + 50k = 25 - 24k^2 - 52k - 24 = 0$$

$$k^2 - 2k = 1 = 0$$

$$(k-1)^2 = 0$$

$$\Rightarrow \quad k - 1 = 0$$

$$k = 1$$

5. Show that the equation $x^2 + (mx + c)^2 = a^2$ has equal roots,
If $c^2 = a^2\,(1 + m^2)$

Solution:

$$x^2 + (mx + c)^2 = a^2$$

$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$

$$(1 + m^2)x^2 + 2mcx + c^2 = a^2$$

$$(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$$

Here $a = 1 + m^2$, $b = 2mc$, $c = c^2 = a^2$

As the roots are equal, so

Disc $= 0$

$$b^2 - 4ac = 0$$
$$(2mc)^2 - 4(1+m^2)(c^2 - a^2) = 0$$
$$4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - a^2m^2) = 0$$
$$4m^2c^2 - 4c^2 + 4a^2 - 4mc^2 + 4a^2m^2 = 0$$
$$-4c^2 + 4a^2 + 4ac^2 + 4a^2m^2 = 0$$
$$-4(c^2 - a^2 - a^2m^2) = 0$$
$$c^2 - a^2 - a^2m^2 = 0$$
$$\Rightarrow \quad c^2 = a^2 + a^2m^2$$
$$c^2 = a^2(a + m^2)$$

Hence proved.

6. Find the condition that the roots of the equation $(my + c)^2 - 4ax = 0$ are equal.

Solution:

$$(mx + c)^2 - 4ax = 0$$
$$m^2x^2 + 2mcx + c^2 - 4ax = 0$$
$$m^2x^2 + 2mcx - 4ax + c^2 = 0$$
$$m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

Here $a = m^2$, $b = 2(mc - 2a)$, $c = c^2$

As the roots are equal, so

$$\text{Disc} = 0$$
$$b^2 - 4ac = 0$$
$$[2(mc - 2a)]^2 - 4(m^2)(c^2) = 0$$
$$4(mc - 2a)^2 - 4m^2c^2 = 0$$
$$4(m^2c^2 - 4amc + 4a^2) - 4m^2c^2 = 0$$
$$4(m^2c^2 - 4amc + 4a^2 - m^2c^2) = 0$$
$$\Rightarrow \quad 4a2 - 4amc = 0$$
$$4a(a - mc) = 0$$
$$\Rightarrow \quad a - mc = 0$$
$$a = mc$$

Which is the required condition.

7. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^3 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.

Solution:

$$\left(c^2 - ab\right)x^2 - 2\left(a^2 - bc\right)x + \left(b^2 - ac\right) = 0$$

Here $a = c^2 - ab$, $b = -2(a^2 - bc)$, $c = b^2 - ac$

As the roots are equal so

$$\text{Disc} = 0$$

$$b^2 - 4ac = 0$$

$$\left[-2\left(a^2 - bc\right)\right]^2 - 4\left(c^2 - ab\right)\left(b^2 - ac\right) = 0$$

$$4\left(a^2 - bc\right)^2 - 4\left(c^2 - ab\right)\left(b^2 - ac\right) = 0$$

$$4\left[\left(a^2 - 2a^2bc + b^2c^2\right) - \left(b^2c^2 - ac^3 - ab^3 + a^2bc\right)\right] = 0$$

$$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc = 0$$

$$\Rightarrow \quad a^4 + ab^3 + ac^3 - 3a^2bc = 0$$

$$a\left(a^3 + b^3 + c^3 - 3abc\right) = 0$$

Either $a = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

$$a^3 + b^3 + c^3 = 3abc$$

Hence proved.

8. Show that the roots of the following equations are rational.

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

Solution:

Here $A = a(b - c)$, $B = b(c - a)$, $C = c(a - b)$

Disc $= B^2 - 4AC$

$= [b(c - a)]^2 - 4[a(b - c)][c(a - b)]$

$= b^2(c - a)^2 - 4ac(b - c)(a - b)$

$= b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ac + bc)$

$= b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2$

$= a^2b^2 + b^2c^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2$

$= (ab)^2 + (bc)^2 + (-2ac)^2 + 2(ab)(bc) + 2(bc)(-2ac) + 2(-2ac)(ab)$

$= (ab + bc - 2ac)^2$

Hence the roots are rational.

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

Here $A = a + 2b$, $B = 2(a + b + c)$, $c = a + 2c$

Disc $= B^2 - 4AC$

$= [2(a + b + c)]^2 - 4(a + 2b)(a + 2c)$

$= 4(a + b + c)^2 - 4(a^2 + 2ac + 2ab + 4bc)$

$= 4[\cancel{a^2} + b^2 + c^2 + \cancel{2ab} + 2bc + \cancel{2ca} - \cancel{a^2} - \cancel{2ac} - \cancel{2ab} - 4bc]$

$= 4[b^2 + c^2 - 2bc]$

$= 4(b - c)^2$

Hence the roots are rational.

9. For all values of k, prove that the roots of the equation

$$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0, k \neq 0 \text{ are real.}$$

Solution:

$$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0$$

Here $a = 1, b = -2\left(k + \frac{1}{k}\right), c = 3$

Disc. $= b^2 - 4ac$

$$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(3)$$

$$= 4\left(k + \frac{1}{k^2}\right)^2 - 12$$

$$= 4\left[\left(k + \frac{1}{k}\right)^2 - 3\right]$$

$$= 4\left[k^2 + \frac{1}{k^2} + 2 - 3\right]$$

$$= 4\left[k^2 + \frac{1}{k^2} - 1\right] > 0$$

Hence the roots are real.

10. Show that the roots of the equation.

$(b - c)x^2 + (c - a)x + (a - b) = 0$ are real.

Solution:

$(b - c)x^2 + (c - a)x + (a - b) = 0$

Here $A = (b - c), B = c - a, c = a - b$

Disc. $= B^2 - 4AC$

$= (c - a)^2 - 4(b - c)(a - b)$

$= c^2 + a^2 - 2ac - 4(ab - b^2 - ac + bc)$

$= a^2 + c^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$

$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ac$

$= (a)^2 + (-2b)^2 + (c)^2 + 2(a)(-2b) + 2(-2b)(c) + 2(a)(c)$

$= (a - 2b + c)^2 > 0$

Hence the roots of the equation are real.

Cube roots of unity and their properties.

The cube roots of unity:

Let a number x be the cube root of unity,

i.e., $x = (1)^{1/3}$

or $x^3 = 1$

$\Rightarrow$ $x^3 - 1 = 0$

$(x^3) - (1)^3 = 0$

$(x - 1)(x^2 + x + 1) = 0$ [using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

Either $x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow$ $x = 1$ or $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Three cube roots of unity are

$1, \dfrac{-1 + i\sqrt{3}}{2}$ and $\dfrac{-1 - i\sqrt{3}}{2}$, where $i = \sqrt{-1}$.

Recognize complex cube roots of unity as ω and ω^2:

The two complex cube roots of unity are $\dfrac{-1 + \sqrt{-3}}{2}$ and $\dfrac{-1 - \sqrt{-3}}{2}$.

If we name anyone of these as w (pronounced as omega), then the other is ω^2.

Properties of cube roots of unity:

(a) Prove that each of the complex cube roots of unity is the square of the other.

Proof:

The complex cube roots of unity are $\dfrac{-1 + \sqrt{-3}}{2}$ and $\dfrac{-1 - \sqrt{-3}}{2}$.

We prove that

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^2 = \frac{-1 - \sqrt{-3}}{2} \quad \text{and} \quad \left(\frac{-1 - \sqrt{-3}}{2}\right)^2 = \frac{-1 + \sqrt{-3}}{2}$$

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^2 = \frac{1 + (-3) - 2\sqrt{-3}}{4} = \frac{-2 - 2\sqrt{-3}}{4} = \frac{2\left(-1 - \sqrt{-3}\right)}{4} = \frac{1 - 1 - \sqrt{-3}}{2}$$

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^2 = \frac{1 + (-3) + 2\sqrt{-3}}{4} = \frac{-2 + 2\sqrt{-3}}{4} = \frac{2\left(-1 + \sqrt{-3}\right)}{4} = \frac{-1 + \sqrt{-3}}{2}$$

Thus, each of the complex cube root of unity is the square of the other, that is,

If $\omega = = \dfrac{-1 + \sqrt{-3}}{2}$, then $\omega^2 = = \dfrac{-1 - \sqrt{-3}}{2}$ and if $\omega = = \dfrac{-1 - \sqrt{-3}}{2}$ then

$$\omega^2 = \frac{-1+\sqrt{-3}}{2}.$$

(b)Prove that the product of three cube roots of unity is one.

Proof:

Three cube roots of unity are

$$1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}$$

The product of cube roots of unity $= (1)\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$

$$= \frac{(-1)^2-\left(\sqrt{-3}\right)^2}{4} = \frac{-1-(-3)}{4} = \frac{1+3}{4} = \frac{4}{4} = 1$$

i.e., $(1)(\omega)(\omega^2) = 1$ or $\omega^3 = 1$

(c) Prove that each complex cube root of unity is reciprocal of the other.

Proof:

We know that $\omega^3 = 1$ $\Rightarrow$ $\omega\,\omega^2 = 1$, so

$$\omega = \frac{1}{\omega^2} \quad \text{or} \quad \omega^2 = \frac{1}{\omega}$$

Thus, each complex cube root of unity is reciprocal of the other.

(d) Prove that the sum of all the cube roots of unity is zero.

i.e., $1 + \omega + \omega^2 = 0$

Proof:

The cube roots of unity are

$$1, \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2}$$

If $\omega = \frac{-1+\sqrt{-3}}{2}$, then $\omega^2 = \frac{-1-\sqrt{-3}}{2}$

The sum of all the roots $= 1 + \omega + \omega^2$

$$= 1 + \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}$$

$$= \frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2} = \frac{0}{2} = 0$$

Thus, $1 + \omega + \omega^2 = 0$

We can easily deduce the following results, that is,

(i) $1 + \omega^2 = -\omega$ (ii) $1 + \omega = -co^2$ (iii) $\omega + \omega^2 = -1$