SOLVED EXERCISE 2.2

1.   Find the cube roots of -1, 8, –27, 64.

Solution:

(i) The tree cube roots of – 1

Let $x^3 = -1$

$(x)^3 + (1)^3 = 0$

$(x+1)(x^2 - x + 1) = 0$

Either $x + 1 = 0$ or $x^2 - x + 1 = 0$

     $x = -1$ Here $a = 1, b = -1, c = 1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(-1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 + \sqrt{-3}}{2} \quad \text{or} \quad x = \frac{1 - \sqrt{-3}}{2}$$

$$= -\left(\frac{-1-\sqrt{-3}}{2}\right) \qquad = -\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$= -\omega^2 \qquad\qquad\qquad = -\omega$$

Three cube roots of –1 are $-1, -\omega, -\omega^2$

(ii) The three cube roots of 8

Let $x^3 = 8$

$x^3 - 8 = 0$

$(x)^3 - (2)^3 = 0$

$(x - 2)(x^2 + 2x + 4) = 0$

Either $x - 2 = 0$ or $x^2 + 2x + 4 = 0$

     $x = 2$ Here $a = 1, b = 2, c = 4$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2\sqrt{4-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2\left(-1 \pm \sqrt{-3}\right)}{2}$$

$$x = \frac{2\left(-1 \pm i\sqrt{3}\right)}{2} \qquad \because i = \sqrt{-1}$$

$$x = \frac{2\left(-1 + i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x = 2\left(-\frac{1 - i\sqrt{3}}{2}\right)$$

$$= 2\omega \qquad\qquad\qquad\qquad = 2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of – 27

Let $x^3 = -27$

$x^3 + 27 = 0$

$(x)^3 - (3)^3 = 0$

$(x+3)(x^2 - 3x + 9) = 0$

Either $x + 3 = 0$ or $x^2 - 3x + 9 = 0$

       $x = -3$ Here $a = 1; b = -3, c = 9$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = \frac{3\left(-1 \pm i\sqrt{3}\right)}{2} \qquad \because i = \sqrt{-1}$$

$$x = \frac{3\left(1 + i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x = 3\left(-\frac{1 - i\sqrt{3}}{2}\right)$$

$$x = \frac{-3\left(-1 - i\sqrt{3}\right)}{2} \qquad \text{or} \qquad = -3\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$$= -3\omega^2 \qquad\qquad\qquad = -3\omega$$

$\therefore$ Three cube roots of -27 are $-3,\ -3\omega,\ -3\omega^2$

(iv) The three cube roots of 64

Let $x^3 = 64$

$x^3 - 64 = 0$

$(x)^3 - (4)^3 = 0$

$(x - 4)(x^2 + 4x + 16) = 0$

Either $x - 3 = 0$ or $x^2 + 4x + 16 = 0$

$\qquad\quad x = 4$ $\qquad\qquad$ Here $a = 1,\ b = 4,\ c = 16$

$\qquad\qquad\qquad\qquad$ Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$x = \frac{4 \pm \sqrt{-48}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{-3}}{2}$$

$$x = \frac{4\left(-1 \pm i\sqrt{3}\right)}{2} \qquad i = \sqrt{-1}$$

$$x = \frac{3\left(-1 + i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x = 4\left(\frac{-1 - i\sqrt{3}}{2}\right)$$

$$= 4\omega \qquad\qquad\qquad\qquad = 4\omega^2$$

Three cube roots of 64 are 4, 4, ω, $4\omega^2$

2. Evaluate

(i) $(1 - \omega - \omega^2)^7$

Solution:

$$
\begin{aligned}
(1 - \omega - \omega^2)^7 &= \left[1 - \left(\omega + \omega^2\right)\right]^7 \\
&= \left[-1 - (-1)\right]^7 \qquad \because \omega + \omega^2 = -1 \\
&= (1 + 1)^7 \\
&= 2^7 = 128
\end{aligned}
$$

(ii) $(1 - 3\omega - 3\omega^2)^5$

Solution:

$$
\begin{aligned}
(1 - 3\omega - 3\omega^2)^5 &= \left[1 - 3\left(\omega + \omega^2\right)\right]^5 \\
&= \left[1 - 3(-1)\right]^5 \qquad \because \omega + \omega^2 = -1 \\
&= (1 + 3)^5 \\
&= 4^5 = 1024
\end{aligned}
$$

(iii) $(9 + 4\omega + 4\omega^2)^3$

Solution:

$$
\begin{aligned}
(9 + 4\omega + 4\omega^2)^3 &= \left[9 + 4\left(\omega + \omega^2\right)\right]^3 \\
&= \left[9 + 4(-1)\right]^3 \qquad \because \omega + \omega^2 = -1 \\
&= (9 - 4)^3 \\
&= 5^3 = 125
\end{aligned}
$$

(iv) $(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2)$

Solution:

$$
\begin{aligned}
&(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2) \\
&= \left[2(1 + \omega) - 2\omega^2\right]\left[3(1 + \omega^2) - 3\omega\right] \\
&= \left[2(-\omega^2) - 2\omega^2\right]\left[3(-\omega) - 3\omega\right] \qquad \because 1 + \omega = -\omega^2
\end{aligned}
$$

$$= \left(-2\omega^2 - 2\omega^2\right)\left(-3\omega - 3\omega\right) \qquad \because 1 + \omega^2 = -\omega$$

$$= \left(-4\omega^2\right)\left(-6\omega\right)$$

$$= \left(-4\right)\left(-6\right)\left(\omega^2 \cdot \omega\right)$$

$$= 24\omega^3$$

$$= 24(1) \qquad \because \omega^3 = 1$$

$$= 24$$

(v) $\left(-1 + \sqrt{-3}\right)^6 + \left(-1 - \sqrt{-3}\right)^6$

Solution:

$$\left(-1 + \sqrt{-3}\right)^6 + \left(-1 - \sqrt{-3}\right)^6$$

$$= (2\omega)^6 + \left(2\omega^2\right)^6 \qquad \because \omega = \frac{-1 + \sqrt{-3}}{2} \quad \text{and} \quad \omega^2 = \frac{-1 - 1\sqrt{-3}}{2}$$

$$= 2^6\left(\omega^6\right) + 2^6\left(\omega^{12}\right) \qquad \qquad 2\omega = -1 + \sqrt{-3} \qquad 2\omega^2 = -1 - \sqrt{-3}$$

$$= 2^6\left[\left(\omega^3\right)^2\right] + 2^6\left[\left(\omega^3\right)^4\right]$$

$$= 2^6\left[(1)^2 + (1)^4\right] \qquad \because \omega^3 = 1$$

$$= 2^6[1 + 1]$$

$$= 2^6 \cdot 2 = 2^{6+1} = 2^7$$

$$= 128$$

(vi) $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^9$

Solution:

$$\left(\frac{-1 + \sqrt{-3}}{2}\right)^9 + \left(\frac{-1 - \sqrt{-3}}{2}\right)^9$$

$$= \omega^9 + \left(2\omega^2\right)^9 \qquad \because \omega = \frac{-1 + \sqrt{-3}}{2} \text{ and } \omega^2 = \frac{-1 - 1\sqrt{-3}}{2}$$

$$= \omega^9 + \omega^{18}$$

$$= \left(\omega^3\right)^3 + \left(\omega^3\right)^6 = (1)^3 + (1)^6 \qquad \because \omega^3 = 1$$

$$= 1 + 1 = 2$$

(vii) $\omega^{37} + \omega^{38} - 5$

Solution:

$\omega^{37} + \omega^{38} - 5$

$= \omega^{37} + \omega^{38} - 5$

$= \omega^{36}.\omega + \omega^{36}.\omega^2 - 5$

$= \left(\omega^3\right)^{12}.\omega + \left(\omega^3\right)^{12}.\omega^2 - 5$

$= (1)^{12}.\omega + (1)^{12}.\omega^2 - 5 \qquad \because \omega^3 = 1$

$= \omega + \omega^2 - 5$

$= -1 - 5 \qquad \because \omega + \omega^2 = -1$

$= -6$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

$\omega^{-13} + \omega^{-17}$

$= \omega^{-13} + \omega^{-17}$

$= \omega^{-12-1} + \omega^{-15-2}$

$= \omega^{12}.\omega^{-1} + \omega^{-15}.\omega^{-2}$

$= \left(\omega^3\right)^{-4}.\omega^{-1} + \left(\omega^3\right)^{-5}.\omega^{-2}$

$= (1)^{-4}.\omega^{-1} + (1)^{-5}.\omega^{-2} \qquad \because \omega^3 = 1$

$= \omega^{-1} + \omega^{-2}$

$= \dfrac{1}{\omega} + \dfrac{1}{\omega^2}$

$= \dfrac{\omega^2 + \omega}{\omega^3} \qquad \because \omega^2 + \omega = -1 \text{ and } \omega^3 = 1$

$= \dfrac{-1}{1}$

$= -1$

3. Prove that $x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$.

Solution:

$x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$

R.H.S. $= (x + y)(x + \omega y)(x + \omega^2 y)$

$= (x + y)\left[x(x + \omega^2 y) + \omega y(\omega + \omega^2 y)\right]$

$= (x + y)\left[x^2 + \omega^2 xy + \omega xy + \omega^3 y^2\right]$

$= (x + y)\left[x^2 + (\omega^2 + \omega)xy + (1)y^2\right] \qquad \because \omega^3 = 1$

$$= (x + y)\left[x^2 + (-1)xy + y^2\right] \qquad \because \omega^2 + \omega = -1$$

$$= (x + y)\left(x^2 - xy + y^2\right)$$

$$= \text{L.H.S.}$$

Hence Proved

4.   Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$.

Solution:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

R.H.S. $= (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

$$= (x + y + z)\left[x(x + \omega^2 y + \omega z) + \omega y(x + \omega^2 y + \omega z) + \omega^2 z(x + \omega^2 y + \omega z)\right]$$

$$= (x + y + z)\left[x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^4 z + \omega^3 z^2\right]$$

$$= (x + y + z)\left[x^2 + \omega^2 xy + \omega xy + \omega^2 yz + \omega yz + \omega^2 xz + \omega xz + (1)y^2 + (1)z^2\right]$$

$$= (x + y + z)\left[x^2 + \left(\omega^2 + \omega\right)xy + \left(\omega^2 + \omega\right)yz + \left(\omega^2 + \omega\right)xz + y^2 + z^2\right] \because \omega^3 = 1$$

$$= (x + y + z)\left[x^2 + (-1)xy + (-1)yz + (-1)xz + y^2 + z^2\right] \quad \because \omega^2 + \omega = -1$$

$$= (x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right)$$

$$= x^3 + y^3 + z^3 - 3xz$$

$$= \text{L.H.S.}$$

Hence proved.

5.   Prove that $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ 2n factors $= 1$.

Solution:

L.H.S. $= (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots$ 2n factors

$$= (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\ldots\ldots\ldots\text{2n factors}$$

$$= (1 + \omega)(1 + \omega^2)(1 + \omega^3.\omega)(1 + \omega^2.\omega^6)\ldots\ldots\text{2n factors}$$

$$= (1 + \omega)(1 + \omega^2)(1 + \omega^3.\omega)\left(1 + \omega^2.(\omega^3)^2\right)\ldots\ldots\text{2n factors}$$

$$= (1 + \omega)(1 + \omega^2)(1 + (1)\omega)\left(1 + \omega^2(1)^2\right)\ldots\ldots\text{2n factors} \quad \because \omega^3 = 1$$

$$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)\ldots\ldots\text{2n factors}$$

$$= (-\omega)^2(-\omega)(-\omega^2)(-\omega)\ldots\ldots\text{2n factors} \qquad \because 1 + \omega = -\omega^2$$

$$= \left[(-\omega)^2(-\omega)\right]\left[(-\omega^2)(-\omega)\right]\ldots\ldots\text{n factors} \qquad \because 1 + \omega^2 = -\omega$$

$$= \left[\omega^3\right]\left[\omega^3\right]\ldots\ldots\text{n factors}$$

$$= (1)(1) \ldots \ldots n \text{ factors} \qquad \qquad \because \omega^3 = 1$$

$$= (1)^n$$

$$= 1$$

$$= \text{R.H.S.}$$

Hence proved.

Roots and co-efficient of a quadratic equation:

We know that $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ are roots of the equation $ax^2 + bx + c = 0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co-efficient of a quadratic equation:

If $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$,

then we can find the sum and the product of the roots as follows.

Sum of the' roots $= \alpha + \beta$

$$= \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \dfrac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{-2b}{2a} = -\dfrac{b}{a}$$

Product of the roots $= \alpha \beta$

$$= \left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \right)\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \right)$$

$$= \dfrac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2} = \dfrac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \dfrac{b^2 - b^2 + 4ac}{4a^2} = \dfrac{4ac}{4a^2} = \dfrac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S = -\dfrac{b}{a} = -\dfrac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2} \quad \text{and} \quad P = \dfrac{c}{a} = \dfrac{\text{Constant term}}{\text{Co-efficient of } x^2}$$

SOLVED EXERCISE 2.3

1. Without solving, find the sum and the product of the following quadratic equations.

(i) $x^2 - 5x + 3 = 0$

Solution: