$$\Rightarrow \qquad n = \frac{2}{3}$$

Find 3rd, 4th, mean and continued proportion:

We are already familiar with proportions that if quantities a, b, c and d are in proportion, then a : b :: c : d

i.e., product of extremes = product of means

Third Proportional

If three quantities a, b and c are related as a : b :: b : c, then c is called the third proportion.

Fourth Proportional

If four quantities a, b, c and d are related as

a : b :: c : d

Then d is called the fourth proportional.

Mean Proportional

If three quantities a, b and c are related as a : b :: b : c,
then b is called the mean proportional.

Continued Proportion

If three quantities a, b and c are related as

a : b :: b : c

where a is first, b is the mean and c is the third proportional, then a, b and c are in continued proportion.

SOLVED EXERCISE 3.3

1. Find a third proportional to

(i) 6, 12

Solution:

Let C be the third proportional, then

$$6 : 12 :: 12 : C$$

∴ Product of extremes = Product of means

$$6C = 12 \times 12$$

$$6C = 144$$

$$C = \frac{144}{6}$$

$$C = 24$$

(ii) $a^2 - b^2, a - b$

Solution:

Let C be the third proportional, then

$$a^3 : 3a^2 :: 3a^2 : C$$

Product of extremes = Product of means

$$(C)(a^3) = (3a^2)(3a^2)$$

$$C = \frac{(3a^2)(3a^2)}{a^3}$$

$$C = \frac{9a^4}{a^3}$$

$$C = 9a$$

$$6C = 144$$

$$C = \frac{144}{6}$$

$$C = 24$$

(iii) $a^2 - b^2, a - b$

Solution:

Let C be the third proportional, then

$$a^2 - b^2 : a - b :: a - b : C$$

$\therefore$ Product of extremes = Product of means

$$(a^2 - b^2)(C) = (a - b)(a - b)$$

$$C = \frac{(a - b)(a - b)}{(a^2 - b^2)}$$

$$C = \frac{(a - b)(a - b)}{(a - b)(a + b)}$$

$$C = \frac{a - b}{a + b}$$

(iv) $(x - y)^2, x^3 - y^3$

Solution:

Let C be the third proportional, then

$$(x - y)^2 : x^3 - y^3 :: x^3 - y^3 : C$$

$\therefore$ Product of extremes = Product of means

$$C(x - y)^2 = (x^3 - y)^3 . (x^3 - y^3)$$

$$C = \frac{(x - y)(x^2 + xy + y^2)(x - y)(x^2 + xy + y^2)}{(x - y)^2}$$

$$C = \frac{(x - y)^2 (x^2 + xy + y)^2}{(x - y)}$$

$$C = (x^2 + xy + y)^2$$

(v) $(x+y)^2, x^2-xy-2y^2$

Solution:

Let C be the third proportional, then

$$(x+y)^2 : x^2-xy-2y^2 :: x^2-xy-2y^2 : C$$

$\therefore$ Product of extremes = Product of means

$$C(x+y)^2(x^2-xy-2y^2)(x^2-xy-2y^2)^2$$

$$C=\frac{(x^2-xy-2y^2)^2}{(x+y)^2}$$

$$C=\frac{[(x-2y)(x+y)]}{(x+y)^2}$$

$$C=\frac{(x-2y)^2(x+y)^2}{(x+y)^2}$$

$$C=(x-2y)^2$$

(vi) $\frac{p^2-q^2}{p^3+q^3}, \frac{p-q}{p^2-pq+q^2}$

Solution:

Let C be the third proportional, then

$$\frac{p^2-q^2}{p^3+q^3} : \frac{p-q}{p^2-pq+q^2} :: \frac{p-q}{p^2-pq+q^2} : C$$

$\therefore$ Product of extremes = Product of means

$$C\left(\frac{p^2-q^2}{p^3+q^3}\right)=\left(\frac{p-q}{p^2-pq+q^2}\right)\left(\frac{p-q}{p^2-pq+q^2}\right)$$

$$C=\frac{(p-q)^2}{(p^2-pq+q^2)}\times\frac{p^3+q^3}{p^2-q^2}$$

$$C=\frac{(p-q)^2}{(p^2-pq+q^2)^2}\times\frac{(p+q)(p^2-pq+q^2)}{(p+q)(p-q)}$$

$$C=\frac{p-q}{p^2-pq+q^2}$$

2. Find a fourth proportional to

(i) 5, 8, 15

Solution:

Let x be the fourth proportional, then

$$5 : 8 :: 15 : x$$

∴ Product of extremes = Product of means

$$(5)(x) = (8)(15)$$

$$x = \frac{(8)(15)}{5}$$

$$x = 8 \times 3 = 24$$

(ii) $4x^4, 2x^3, 18x^5$

Solution:

Let C be the fourth proportional, then

$$4x^4 : 2x^3 :: 18x^5 : C$$

∴ Product of extremes = Product of means

$$(4x^4)(C) = (2x^3)(18x^5)$$

$$C = \frac{36x^8}{4x^4}$$

$$C = 9x^4$$

(iii) $15a^5b^6, 10a^2b^5, 21a^3b^3$

Solution:

Let x be the fourth proportional, then

$$15a^5b^6 : 10a^2b^5 :: 21a^3b^3 : x$$

∴ Product of extremes = Product of means

$$x(15a^5b^6) = (10a^2b^5)(21a^3b^3)$$

$$x = \frac{10 \times 21a^5b^8}{15a^5b^6}$$

$$x = 2 \times 7b^2$$

$$x = 14b^2$$

(iv) $x^2 - 11x + 24, (x - 3), 5x^4 - 40x^3$

Solution:

Let C be the fourth proportional, then

$$x^2 - 11x + 24 : (x - 3) :: 5x^4 - 40x^3 : C$$

∴ Product of extremes = Product of means

$$C\left(x^2 - 11x + 24\right) = (x-3)\left(5x^4 - 40x^3\right)$$

$$C = \frac{(x-3)\left[5x^3(x-8)\right]}{x^2 - 11x + 24}$$

$$C = \frac{5x^3(x-3)(x-8)}{(x-3)(x-8)}$$

$$C = 5x^3$$

(v) $p^3 + q^3, p^2 - q^2, p^2 - pq + q^2$

Solution:

Let C be the fourth proportional, then

$$p^3 + q^3 : p^2 - q^2 :: p^2 - pq + q^2 : C$$

$\therefore$ Product of extremes = Product of means

$$C\left(p^3 + q^3\right) = \left(p^2 - q^2\right)\left(p^2 - pq + q^2\right)$$

$$C = \frac{(p-q)(p+q)\left(p^2 - pq + q^2\right)}{\left(p^3 + q^3\right)}$$

$$C = \frac{(p-q)\left(p^3 + q^3\right)}{p^3 + q^3}$$

$$C = p - q$$

(vi) $\left(p^2 - q^2\right)\left(p^2 + pq + q^2\right), p^3 + q^3, p^3 - q^3$

Solution:

Let x be the fourth proportional, then

$$\left(p^2 - q^2\right)\left(p^2 + pq + q^2\right) : p^3 + q^3 :: p^3 - q^3 : x$$

$\therefore$ Product of extremes = Product of means

$$x\left(p^2 - q^2\right) = \left(p^2 + pq + q^2\right) = \left(p^3 + q^3\right)\left(p^3 - q^3\right)$$

$$x = \frac{\left(p^3 + q^3\right)\left(p^3 - q^3\right)}{(p+q)(p-q)\left(p^2 + pq + q^2\right)}$$

$$= \frac{(p+q)\left(p^2 - pq + q^2\right)\left(p^3 - q^3\right)}{(p+q)\left(p^3 - q^3\right)}$$

$$= p^2 - pq + q^2$$

3. Find a mean proportional between

(i) 20, 45

Solution:

Let m be the mean proportional, then

$$20 : m :: m : 45$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 20 \times 45$$

$$m^2 = 900$$

$$m = \pm\sqrt{900}$$

$$m = \pm 30$$

(ii) $20x^3y^5, 5x^7y$

Solution:

Let m be the mean proportional, then

$$20x^3y^5 : m :: m : 5x^7y$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 20x^3y^5 \times 5x^7y$$

$$m^2 = 100x^{10}y^6$$

$$= \pm\sqrt{100x^{10}y^6}$$

$$= \pm\left(10x^{10}y^6\right)^{1/2}$$

$$= \pm\left(10^2\right)^{1/2}\left(x^{10}\right)^{\frac{1}{2}}\left(y^6\right)^{\frac{1}{2}}$$

$$= \pm 10x^5y^3$$

(iii) $15p^4qr^3, 135q^5r^7$

Solution:

Let m be the mean proportional, then

$$15p^4qr^3 : m :: m : 135q^5r^7$$

$\therefore$ Product of means = Product of extremes

$$m \times m = 15p^4qr^3 \times 135q^5r^7$$

$$m^2 = 15 \times 135p^4q^6r^{10}$$

$$m^2 = 2025p^4q^6r^{10}$$

$$m = \pm\sqrt{2025p^4q^6r^{10}}$$

$$m = \pm\left(45^2\right)^{\frac{1}{2}}\left(p^4\right)^{\frac{1}{2}}\left(q^6\right)^{\frac{1}{2}}\left(r^{10}\right)^{\frac{1}{2}}$$

$$m = \pm 45p^2q^3r^5$$

(iv) $x^2 - y^2, \dfrac{x-y}{x+y}$

Solution:

Let m be the mean proportional, then

$$x^2 - y^2 : m :: m \frac{x-y}{x+y}$$

$\therefore$ Product of means = Product of extremes

$$m \times m = x^2 - y^2 \times \frac{x-y}{x+y}$$

$$m^2 = (x-y)(x+y) \times \frac{x-y}{x+y}$$

$$m^2 = (x-y)(x-y)$$

$$m^2 = (x-y)^2$$

$$m = \pm\sqrt{(x-y)^2}$$

$$m = \pm x - y$$

4. Find the values of the letter involved in the following continued proportions.

(i) 5, p, 45

Solution:

Since 5, P and 45 are in continued proportions.

$$5 : P :: P : 45$$

$\therefore$ Product of means = Product of extremes

$$P \times P = 5 \times 45$$

$$P^2 = 225$$

$$P = \pm\sqrt{225}$$

$$P = \pm 15$$

(ii) 8, x, 18

Solution:

Since 8, x and 18 are in continued proportions.

$$8 : x :: x : 18$$

$\therefore$ Product of means = Product of extremes

$$x \times x = 8 \times 18$$

$$x^2 = 144$$

$$x = \pm\sqrt{144}$$

$$x = \pm 12$$

(iii) 12, 3p – 6,27

Solution:

Since 12, 3P – 6 and 27 are in continued proportions.

$$12 : 3P - 6 :: 3P - 6 : 27$$

∴ Product of means = Product of extremes

$$(3p-6)(3p-6) = 12 \times 27$$

$$(3p-6)^2 = 324$$

$$\sqrt{(3p-6)^2} = \pm\sqrt{324}$$

$$3P - 6 = \pm 18$$

$\Rightarrow \quad 3p - 6 = -18 \quad$ or $\quad 3p - 6 = 18$

$3p = 6 - 18 \qquad 3p = 18 + 6$

$3p = -12 \qquad 3p = 24$

$\Rightarrow \quad p = -4 \qquad \Rightarrow \quad p = 8$

(iv) 7, m – 3, 28

Solution:

Since 7, m – 3, 28 and 45 are in continued proportions.

$$7 : m-3 :: m-3 : 28$$

∴ Product of means = Product of extremes

$$(m-3).(m-3) = 7 \times 28$$

$$(m-3)^2 = 196$$

$$\sqrt{(m-3)^2} = \pm\sqrt{196}$$

$$m - 3 = \pm 14$$

$\Rightarrow \quad m - 3 = -14 \quad$ or $\quad m - 3 = 14$

$m = 3 - 14 \qquad m = 3 + 14$

$m = -11 \qquad m = 17$

Theorems on Proportions:

If four quantities a, b, c and d form a proportion, then many other useful properties may be deduced by the properties of fractions.

(1) Theorem of Invertendo

If a : b = c : d, then b : a = d : c

(2) Theorem of Alternando

If a : b = c : d, then a : c = b : d

(3) Theorem of Componendo

If a : b = c : d, then

(i) a + b : b = c + d : d

and (ii) a : a + b = c : c + d

(4) Theorem of Dividendo

If a : b = c : d, then

(i) a – b : b = c – d : d

and (ii) a : a − b = c : c − d

(5) Theorem of Componendo dividendo

If a : b = e : d, then

(i) a + b : a − b = c + d : c − d

and (ii) a − b : a + b = c − d : c + d

SOLVED EXERCISE 3.4

1. Prove that a : b = c : d, if

(i) $$\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$$

Solution:

Given $$\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$$

By componendo-dividendo theorem, we have

$$\frac{(4a+5b)+(4c+5b)}{(4a-5b)-(4c-5b)}=\frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a+5b+4a-5b}{4a+5b-4a+5b}=\frac{4c+5d+4c-5d}{4c+5d-4c-5d}$$

$$\frac{8a}{10b}=\frac{8c}{10d}$$

Multiplying both sides by $\frac{18}{4}$, we get

$$\frac{a}{b}=\frac{c}{d}$$

⇒ a : b = c : d

Hence proved

(ii) $$\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$$

Solution:

Given

$$\frac{2a+9b}{2a-9b}=\frac{2c+9d}{2c-9d}$$

By componendo-dividendo theorem, we have