

Rule I:

If linear factor $(ax + b)$ occurs as a factor of $D(x)$, then there is a partial fraction of the

form $\frac{A}{ax + b}$, where A is a constant to be found.

In $\frac{N(x)}{D(x)}$, the polynomial $D(x)$ may be written as,

$D(x) = (a_1x + b)(a_2x + b_2)\dots(a_nx + b_n)$ with all factors distinct.

We have, $\frac{N(x)}{D(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3}{a_3x + b_3} + \dots + \frac{A_n}{a_nx + b_n}$,

where A_1, A_2, \dots, A_n are constants to-be determined.

Note:

General method applicable to resolve all rational fractions of the form $\frac{N(x)}{D(x)}$ is as follows:

- (i) The numerator $N(x)$ must be of lower degree than the denominator $D(x)$.
- (ii) If degree of $N(x)$ is greater than the degree of $D(x)$, then division is used and the remainder fraction $R(x)$ can be broken into partial fractions.
- (iii) Make substitution of constants accordingly
- (iv) Multiply both the sides by L.C.M.
- (v) Arrange the terms on both sides in descending order.
- (vi) Equate the coefficients of like powers of x on both sides, we get as many as equations as there are constants in assumption.
- (vii) Solving these equations, we can find the values of constants.

SOLVED EXERCISE 4.1

Resolve into partial fractions.

$$(1) \frac{7x - 9}{(x + 1)(x - 3)}$$

Solution:

$$\frac{7x - 9}{(x + 1)(x - 3)} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

Multiplying both sides by $(x + 1)(x - 3)$, we get

$$7x - 9 = A(x - 3) + B(x + 1) \quad (1)$$

To find A , we put $x = 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$7(1) - 9 = A(-1 - 3) + B(-1 + 1)$$

$$7 - 9 = A(-4) + B(0)$$

$$= 16 = -4A$$

$$-4A = 2 - 16$$

Dividing both sides '-4', we get

$$A = -4$$

To find B, we put $x - 3 = 0 \Rightarrow x = 3$ in eq. (1), we get

$$7(3) - 9 = A(3 - 3) + b(3 + 1)$$

$$21 - 9 = A(0) + B(4)$$

$$12 = 4B$$

Or $4B = 12$

Dividing both sides by '4', we get

$$B = 3$$

Thus required partial fractions are $\frac{-4}{x+1} + \frac{3}{x-3}$

$$\text{Hence, } \frac{7x-9}{(x+1)(x-3)} = \frac{-4}{x+1} + \frac{3}{x-3}$$

$$(2) \frac{x-11}{(x-4)(x+3)}$$

Solution:

$$\frac{x-11}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

Multiplying both sides by $(x-4)(x+3)$, we get

$$x - 11 = A(x + 3) + B(x - 4) \quad (1)$$

To find A, we put $x - 4 = 0 \Rightarrow x = 4$ in eq. (1), we get

$$4 - 11 = A(4 + 3) + B(4 - 4)$$

$$-7 = A(7) + B(0)$$

$$-7 = 7A$$

or $7A = -7$

Dividing both sides by '7', we get

$$A = -1$$

To find B, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$-3 - 11 = A(-3 + 3) + B(-3 - 4)$$

$$-14 = A(0) + B(-7)$$

$$-14 = -7B$$

Or $-7B = -14$

Dividing both sides by '-7', we get

$$B = 2$$

Thus required partial fractions are $\frac{-1}{x-4} + \frac{2}{x+3}$

$$\text{Hence, } \frac{x-11}{(x-4)(x+3)} = \frac{-1}{x-4} + \frac{2}{x+3}$$

$$(3) \frac{3x-1}{x^2-1}$$

Solution:

$$\frac{3x-1}{x^2-1} = \frac{3x-1}{(x-1)(x+1)}$$

$$\text{Let } \frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Multiplying both sides by $(x-1)(x+1)$, we get

$$3x-1 = A(x+1) + B(x-1) \quad (1)$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$3(1)-1 = A(1+1) + B(1-1)$$

$$3-1 = A(2) + B(0)$$

$$2 = 2A$$

$$\text{or } 2A = 2$$

Dividing both sides by '2', we get

$$A = 1$$

To find B, we put $x+1=0 \Rightarrow x=-1$ in eq. (1), we get

$$3(-1)-1 = A(-1+1) + B(-1-1)$$

$$-3-1 = A(0) + B(-2)$$

$$-4 = -2B$$

$$\text{Or } -2B = -4$$

Dividing both sides by '-2', we get

$$B = 2$$

Thus required partial fractions are $\frac{-1}{x-1} + \frac{2}{x+1}$

$$\text{Hence, } \frac{3x-1}{x^2-1} = \frac{-1}{x-1} + \frac{2}{x+1}$$

$$(4) \frac{x-5}{x^2+2x-3}$$

Solution:

$$\frac{x-5}{x^2+2x-3} = \frac{x-5}{x^2+3x-x-3}$$

$$= \frac{x-5}{x(x+3)-1(x+3)} = \frac{x-5}{(x-1)(x+3)}$$

$$\text{Let } \frac{x-5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

Multiplying both sides by $(x-1)(x+3)$, we get

$$x-5 = A(x+3) + B(x-1) \quad (1)$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$1-5 = A(1+3) + B(1-1)$$

$$-4 = A(1+3) + B(0)$$

$$-4 = 4A$$

$$\text{Or } 4A = -4$$

Dividing both sides by '4', we get

$$A = -1$$

To find B, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$-3 - 5 = A(-3 + 3) + B(-3 - 1)$$

$$-8 = A(0) + B(-4)$$

$$-8 = -4B$$

Or $-4B = -8$

Dividing both sides by '-4', we get

$$B = 2$$

Thus required partial fractions are $\frac{-1}{x-1} + \frac{2}{x+3}$

Hence, $\frac{x-5}{x^2+2x-3} = -\frac{1}{x-1} + \frac{2}{x+3}$

(5) $\frac{3x+3}{(x-1)(x+2)}$

Solution:

Let $\frac{3x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Multiplying both sides by $(x-1)(x+2)$, we get

$$3x+3 = A(x+2) + B(x-1) \quad \text{--- (1)}$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$3(1)+3 = A(1+2) + B(1-1)$$

$$3+3 = A(3) + B(0)$$

$$6 = 3A$$

Or $3A = 6$

Dividing both sides by '3', we get

$$A = 2$$

To find B, we put $x+2=0 \Rightarrow x=-2$ in eq. (1), we get

$$3(2)+3 = A(-2+2) + B(-2-1)$$

$$-6+3 = A(0) + B(-3)$$

$$-3 = -3B$$

Or $-3B = -3$

Dividing both sides by '-3', we get

$$B = 1$$

Thus required partial fractions are $\frac{2}{x-1} + \frac{1}{x+2}$

Hence, $\frac{3x+3}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{1}{x+2}$

$$(6) \frac{7x - 25}{(x - 4)(x - 3)}$$

Solution:

$$\text{Let } \frac{7x - 25}{(x - 4)(x - 3)} = \frac{A}{x - 4} + \frac{B}{x - 3}$$

Multiplying both sides by $(x - 4)(x - 3)$, we get

$$7x - 25 = A(x - 3) + B(x - 4) \quad (1)$$

To find A, we put $x - 4 = 0 \Rightarrow x = 4$ in eq. (1), we get

$$7(4) - 25 = A(4 - 3) + B(4 - 4)$$

$$28 - 25 = A(1) + B(0)$$

$$3 = A$$

$$\text{Or } A = 3$$

To find B, we put $x - 3 = 0 \Rightarrow x = 3$ in eq. (1), we get

$$7(3) - 25 = A(3 - 3) + B(3 - 4)$$

$$21 - 25 = A(0) + B(-1)$$

$$-B = -4$$

$$\text{Or } B = 4$$

$$B = 4$$

Thus required partial fractions are $\frac{3}{x - 4} + \frac{4}{x - 3}$

$$\text{Hence, } \frac{7x - 25}{(x - 4)(x - 3)} = \frac{3}{x - 4} + \frac{4}{x - 3}$$

$$(7) \frac{x^2 + 2x + 1}{(x - 2)(x + 3)}$$

Solution:

$$\frac{x^2 + 2x + 1}{(x - 2)(x + 3)} = \frac{x^2 + 2x + 1}{x^2 + 3x - 2x - 6}$$

$$= \frac{x^2 + 2x + 1}{x^2 + x - 6}$$

By long division, we have

$$\begin{array}{r} \overline{) x^2 + 2x + 1} \\ \underline{x^2 + x - 6} \\ 7 \end{array}$$

$$\frac{x^2 + 2x + 1}{(x - 2)(x + 3)} = 1 + \frac{x + 7}{x^2 + x - 6}$$

$$= 1 + \frac{x+7}{(x-2)(x+3)}$$

Let $\frac{x+7}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$

Multiplying both sides by $(x-2)(x+3)$, we get

$$x+7 = A(x+3) + B(x-2) \quad (1)$$

To find A, we put $x-2=0 \Rightarrow x=2$ in eq. (1), we get

$$2+7 = A(2+3) + B(2-2)$$

$$9 = A(5) + B(0)$$

$$9 = 5A$$

Or $5A = 9$

Dividing both sides by '5', we get

$$A = \frac{9}{5}$$

To find B, we put $x+3=0 \Rightarrow x=-3$ in eq. (1), we get

$$-3+7 = A(-3+3) + B(-3-2)$$

$$4 = A(0) + B(-5)$$

$$4 = -5B$$

Or $-5B = 4$

Dividing both sides by '-5', we get

$$B = \frac{4}{5}$$

Thus required partial fractions are $\frac{9/5}{x-2} + \frac{-4/5}{x+3}$

Hence, $\frac{x^2+2x+1}{(x-2)(x+3)} = 1 + \frac{9}{5(x-2)} - \frac{4}{5(x+3)}$

(8) $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution:

By long division, we have

$$\begin{array}{r} 2x+3 \\ 3x^2-2x-1 \overline{) 6x^3+5x^2-7} \\ \underline{\pm 6x^3 \mp 4x^2 \mp 2x} \\ 9x^2+2x-7 \\ \underline{\pm 9x^2 \mp 6x \mp 3} \\ 8x-4 \end{array}$$

$$\begin{aligned}\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} &= 2x + 3 + \frac{8x - 4}{3x^2 - 3x + x - 1} \\ &= 2x + 3 + \frac{8x - 4}{3x(x-1) + 1(x-1)} \\ &= 2x + 3 + \frac{8x - 4}{(3x+1)(x-1)}\end{aligned}$$

Let $\frac{8x - 4}{(3x+1)(x-1)} = \frac{A}{3x+1} + \frac{B}{x-1}$

Multiplying both sides by $(3x+1)(x-1)$, we get

$$8x - 4 = A(x-1) + B(3x+1) \quad (1)$$

To find A, we put $3x+1=0 \Rightarrow 3x=-1 \Rightarrow x=-\frac{1}{3}$ in eq. (1), we get

$$8\left(-\frac{1}{3}\right) - 4 = A\left(-\frac{1}{3} - 1\right) + B\left[3\left(-\frac{1}{3}\right) + 1\right]$$

$$-\frac{8}{3} - 4 = A\left(-\frac{4}{3}\right) + B(0)$$

$$-\frac{20}{3} = -\frac{4}{3}A$$

Or $-\frac{4}{3}A = -\frac{20}{3}$

$$\Rightarrow \frac{4}{3}A = \frac{20}{3}$$

$$A = \frac{20}{3} \times \frac{3}{4}$$

$$A = 5$$

To find B, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$8(1) - 4 = A(1-1) + B[3(1)+1]$$

$$8 - 4 = A(0) + B(4)$$

$$4 = 4B$$

Or $4B = 4$

Dividing both sides by '4', we get

$$B = 1$$

Thus required partial fractions are $\frac{5}{3x+1} + \frac{1}{x-1}$

Hence, $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{5}{3x+1} + \frac{1}{x-1}$

Resolution of a fraction when D(x) consists of repeated linear factors:

Rule II:

If a linear factor $(ax + b)$ occurs n times as a factor of $D(x)$, then there are n partial fractions of the form.

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_n}{(ax + b)^n} \text{ where } A_1, A_2, \dots, A_n, \text{ are constants and } n \geq 2 \text{ is a}$$

positive integer.

$$\frac{N(x)}{D(x)} = \frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_n}{(ax + b)^n}$$