$\Rightarrow \quad A = \frac{1}{4}$

To find B, we put $(x+1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$1 = A(-1+1)^2 + B(-1-1)(-1+1) + C(-1-1)$$

$$1 = A(0)^2 + B(-2)(0) + C(-2)$$

$$1 = A(0) + B(0) + C(-2)$$

$$1 = -2C$$

Or $\quad -2C = 1$

$\Rightarrow \quad C = -\frac{1}{2}$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$A + B = 0$

$\frac{1}{4} + B = 0$

$B = -\frac{1}{4}$

Thus required partial fractions are $\frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$

Hence, $\frac{1}{(x-1)(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$

Resolution of fraction when D (x) consists of non-repeated irreducible quadratic factors.

Rule III:

If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$ occur once as a factor of D(x), the partial fraction is of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where A and B are constants to be found.

SOLVED EXERCISE 4.3

Resolve into partial fractions.

(1) $\frac{3x-11}{(x+3)(x^2+1)}$

Solution:

$$\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$$3x-11=A(x^2+1)+(Bx+C)(x+3) \qquad \_\_\_\_\_\_ (1)$$

$$3x-11=Ax^2+A+Bx^2+3Bx+Cx+3C$$

$$3x-11=Ax^2+Bx^2+3Bx+Cx+A+3C \qquad \_\_\_\_\_\_ (2)$$

To find A, we put $x+3=0 \Rightarrow x=-3$ in eq. (1), we get

$$3(-3)11=A((-3)^2+1)+(B(-3)+C)(-3+3)$$

$$-9-11=A(9+1)+(-3B+C)(0)$$

$$-20=10A$$

or $\quad 10A=-20$

Dividing both sides by '10', we get

$$A=-2$$

To find B and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$$A+B=0$$
$$-2+B=0$$
$$B=6$$
$$B=2$$

And $\quad A+3C=-11$

$$-2+3C=-11$$
$$3C=-11+2$$
$$3C=-9$$

Dividing both sides by '3', we get

$$C=-3$$

Thus required partial fractions are $\dfrac{-2}{x+3}+\dfrac{2x+(-3)}{x^2+1}$

Hence, $\dfrac{3x-11}{(x+3)(x^2+1)}=\dfrac{-2}{x+3}+\dfrac{2x-3}{x^2+1}$

(2) $\dfrac{3x+7}{(x^2+1)(x+3)}$

Solution:

$$\frac{3x+7}{(x^2+1)(x+3)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+3}$$

Multiplying both sides by $(x^2+1)(x+3)$, we get

$$3x-+7=A(Ax+B)(x+3)+C(x^2+1) \qquad \_\_\_\_\_\_ (1)$$

$$3x-+7=Ax^2+3Ax+Bx+3B+Cx^2+C$$

$$3x+7=Ax^2+Cx^2+3Ax+Bx+3B+C \qquad \_\_\_\_\_\_ (2)$$

To find A, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$3(-3)+7=\left(A(-3)^2+B\right)+(-3+3)+C\left((-3)^2+1\right)$$

$$-9+7=(-3A+B)+(0)+C(9+1)$$

$$-2=10C$$

or $\quad 10C=-2$

Dividing both sides by '10', we get

$$C=-\frac{1}{5}$$

To find A and B, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$$A+C=0$$

$$A+\left(-\frac{1}{5}\right)=0$$

$$A=\frac{1}{5}$$

And $\quad 3B+C=7$

$$3B+\left(-\frac{1}{5}\right)=7$$

$$3B=7+\frac{1}{5}$$

$$3\,B=\frac{36}{5}$$

$$B=\frac{36}{5}\times\frac{1}{3}$$

$$B=\frac{12}{5}$$

Thus required partial fractions are $\frac{1/5x+12/5}{x^2+1}+\frac{-1/5}{x+3}$

Hence, $\frac{3x-7}{(x^2+1)(x+3)}=\frac{x+12}{5(x^2+1)}-\frac{1}{5(x+1)}$

(3) $\frac{1}{(x+1)(x^2+1)}$

Solution:

$$\frac{1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+c}{x^2+1}$$

Multiplying both sides by $(x+1)(x^2+1)$,, we get

$$1 = A(x^2+1)+(Bx+C)(x+1) \quad \text{\_\_\_\_\_\_\_\_ (1)}$$

$$1 = Ax^2+A+Bx^2+Bx+Cx+C$$

$$1 = Ax^2+Bx^2+Bx+Cx+A+C \quad \text{\_\_\_\_\_\_\_\_ (2)}$$

To find A, we put $x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$1 = A((-1)^2+1)+(B(-1)+C)(-1+1)$$

$$1 = A(1+1)+(-B+C)(0)$$

$$1 = A(2)$$

or $\quad 2A = 1$

$\Rightarrow \quad A = \frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$$A + B = 0$$

$$\frac{1}{2} + B = 0 \qquad \because A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

And $\quad A + C = 1$

$$\frac{1}{2} + C = 1 \qquad \because A = \frac{1}{2}$$

$$C = 1 - \frac{1}{2}$$

$$C = \frac{1}{2}$$

Thus required partial fractions are $\frac{1/2}{x+1} + \frac{-1/2x+1/2}{(x^2+1)}$

Hence, $\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} - \frac{x-1}{2(x+1)}$

(4) $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

Let $\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$

Multiplying both sides by $(x+3)(x^2+1)$, we get

$$9x-7=A(x^2+1)+(Bx+C)(x+3) \quad \text{\_\_\_\_\_\_\_ (1)}$$

$$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$$

$$9x-7=Ax^2+Bx^2+3Bx+Cx+A+3C \quad \text{\_\_\_\_\_\_\_ (2)}$$

To find A, we put $x+3=0 \Rightarrow x=-3$ in eq. (1), we get

$$9(-3)-7=A\left((-1)^2+1\right)+\left(B(-1)+C\right)(-1+1)$$

$$-27-7=A(9+1)+(-B+C)(0)$$

$$-34=10A$$

or $$10A=-34$$

$$\Rightarrow \quad A=-\frac{34}{10}=-\frac{17}{5}$$

To find B and C, equating coefficient of x^2 on both sides of eq. (2), we get

$$A+B=0$$

$$-\frac{17}{5}+B=0 \qquad \because A=-\frac{17}{5}$$

$$B=\frac{17}{2}$$

And $$A+3C=-7$$

$$-\frac{17}{5}+3C=-7 \qquad \because A=-\frac{17}{5}$$

$$3C=-7+\frac{17}{5}$$

$$3C=-\frac{18}{5}$$

$$C=-\frac{18}{5}\times\frac{1}{3}$$

$$C=-\frac{6}{5}$$

Thus required partial fractions are $\dfrac{-17/5}{x+1}+\dfrac{17/5x-6/5}{x^2+1}$

Hence, $\dfrac{9x-7}{(x+3)(x^2+1)}=-\dfrac{17}{5(x+1)}+\dfrac{17x-6}{5(x^2+1)}$

(5) $\dfrac{3x+7}{(x+3)(x^2+4)}$

Solution:

Let $\dfrac{3x+7}{(x+3)(x^2+4)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+3)(x^2+4)$, we get

$3x+7 = A(x^2+4) + (Bx+C)(x+3)$ ________ (1)

$3x+7 = Ax^2 + 4A + Bx^2 + 3Bx + Cx + 3C$

$3x+7 = Ax^2 + Bx^2 + 3Bx + Cx + 4A + 3C$ ________ (2)

To find A, we put x + 3 = 0 ⇒ x = − 3 in eq. (1), we get

$$3(-3)+7 = A((-3)^2+4) + (B(-3)+C)(-3+3)$$

$$-9+7 = A(9+4) + (-3B+C)(0)$$

$$-2 = 13A$$

or $13A = -2$

⇒ $A = -\dfrac{3}{13}$

To find B and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$A + B = 0$

$-\dfrac{2}{13} + B = 0$ $\quad\because A = -\dfrac{2}{13}$

$B = \dfrac{2}{13}$

And $4A + 3C = 7$

$4\left(-\dfrac{2}{13}\right) + 3C = 7$ $\quad\because A = -\dfrac{2}{13}$

$-\dfrac{8}{13} + 3C = 7$

$3C = 7 + \dfrac{18}{5}$

$3C = \dfrac{99}{13}$

⇒ $C = \dfrac{99}{13} \times \dfrac{1}{3}$

$C = \dfrac{33}{13}$

Thus required partial fractions are $\dfrac{-2/13}{x+3} + \dfrac{2/13x + 33/13}{x^2+4}$

Hence, $\frac{9x+7}{(x+3)(x^2+4)} = \frac{-2}{13(x+3)} + \frac{2x+336}{13(x^2+4)}$

(6) $\frac{x^2}{(x+2)(x^2+4)}$

Solution:

Let $\frac{x^2}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

Multiplying both sides by $(x+2)(x^2+4)$, we get

$x^2 = A(x^2+4) + (Bx+C)(x+2)$ ________ (1)

$x^2 = Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C$

$x^2 = Ax^2 + Bx^2 + 2Bx + Cx + 4A + 2C$ ________ (2)

To find A, we put $x + 2 = 0 \Rightarrow x = -2$ in eq. (1), we get

$$(-2)^2 = A((-2)^2+4) + (B(-2)+C)(-2+2)$$

$$4 = A(4+4) + (-2B+C)(0)$$

$$4 = 8A$$

or $8A = 4$

$\Rightarrow$ $A = \frac{4}{8} = \frac{1}{2}$

To find B and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$A + B = 1$

$\frac{1}{2} + B = 1$ $\quad \because A = -\frac{1}{2}$

$B = 1 - \frac{1}{2}$

$B = \frac{1}{2}$

And $4A + 2C = 0$

$4\left(\frac{1}{2}\right) + 2C = 0$ $\quad \because A = \frac{1}{2}$

$x + 2C = 0$

$2C = -2$

$\Rightarrow$ $C = -1$

Thus required partial fractions are $\frac{1/2}{x+2} + \frac{1/2x-1}{x^2+4}$

Hence, $\frac{x^2}{(x+2)(x^2+4)} = \frac{1}{2(x+2)} + \frac{x-2}{13(x^2+4)}$

(7) $\frac{1}{x^3+1}$

Solution:

$$\frac{1}{x^3+1} = \frac{1}{(x)^3+(1)^3} = \frac{1}{(x+1)(x^2-x+1)}$$

Let $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$ we get

$$1 = A(x^2-x+1)+(x+1)(Bx+C) \qquad \_\_\_\_\_\_ (1)$$

$$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$1 = Ax^2 + Bx^2 - Ax + Bx + Cx + A + C \qquad \_\_\_\_\_\_ (2)$$

To find A, we put $x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$1 = A((-1)^2 - (-1) + 1) + (B(-1)+C)(-1+1)$$

$$1 = A(1+1+1) + (-B+C)(0)$$

$$1 = A(3) + (-B+C)(0)$$

or $\quad 3A = 1$

$\Rightarrow \quad A = \frac{1}{3}$

To find B, and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$$A + B = 0$$

$$\frac{1}{3} + B = 0 \qquad \because A = -\frac{1}{3}$$

$$B = 1 - \frac{1}{3}$$

And $\quad A + C = 1$

$$\frac{1}{3} + C = 1$$

$$C = 1 - \frac{1}{3}$$

$$C = \frac{2}{3}$$

Thus required partial fractions are $\frac{1/3}{x+1}+\frac{1/3x+2/3}{x^2-x+1}$

Hence, $\frac{1}{x^3+1}=\frac{1}{2(x+1)}-\frac{x-2}{3(x^2-x+1)}$

(8) $\frac{x^2+1}{x^3+1}$

Solution:

$$\frac{x^2+1}{x^3+1}=\frac{x^2+1}{(x)^3+(1)^3}=\frac{x^2+1}{(x+1)(x^2-x+1)}$$

Let $\frac{x^2+1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$

Multiplying both sides by $(x+1)(x^2-x+1)$ we get

$$x^2+1=A(x^2-x+1)+(Bx+C)(x+1) \quad \_\_\_\_\_\_\_ (1)$$

$$x^2+1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$$

$$x^2+1=Ax^2+Bx^2-Ax+Bx+Cx+A+C \quad \_\_\_\_\_\_\_ (2)$$

To find A, we put x + 1 = 0 ⇒ x = – 1 in eq. (1), we get

$$(-1)^2+1=A((-1)^2-(-1)+1)+(B(-1)+C)(-1+1)$$

$$1+1=A(1+1+1)+(-B+C)(0)$$

$$2=A(3)$$

$$\Rightarrow \quad A=\frac{2}{3}$$

To find B, and C, equating coefficient of x^2 and constant on both sides of eq. (2), we get

$$A+B=1$$

$$\frac{2}{3}+B=1 \qquad \because A=-\frac{2}{3}$$

$$B=1-\frac{2}{3}$$

$$B=\frac{1}{3}$$

And $\quad A+C=1$

$$\frac{2}{3}+C=1 \qquad \because A=\frac{2}{3}$$

$$C=1-\frac{2}{3}$$

$$C=\frac{1}{3}$$

Thus required partial fractions are $\frac{2/3}{x+1}+\frac{1/3x+1/3}{x^2-x+1}$

Hence, $\frac{x^2+1}{x^3+1}=\frac{2}{3(x+1)}+\frac{x+1}{3(x^2-x+1)}$

Resolution of a fraction when D (x) has repeated irreducible quadratic factors.

Rule IV:
If a quadratic factor (ax^1+bx+c) with a $\neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

SOLVED EXERCISE 4.4

Resolve into partial fractions.

(1) $\frac{x^3}{(x^2+4)^2}$

Solution:

Let $\frac{x^3}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$

Multiplying both sides by $(x^2+4)^2$, we get

$$x^3=(Ax+B)(x^2+4)+Cx+D \qquad \text{\_\_\_\_\_\_ (1)}$$

$$x^3=Ax^3+4Ax+Bx^2+4B+Cx+D$$

To find A, B, C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq. (2),

We get.

Coefficient of x^1: $\quad A=1$

Coefficient of x^2: $\quad B=0$

Coefficient of x: $\quad 4A+C=0$ ______ (2)