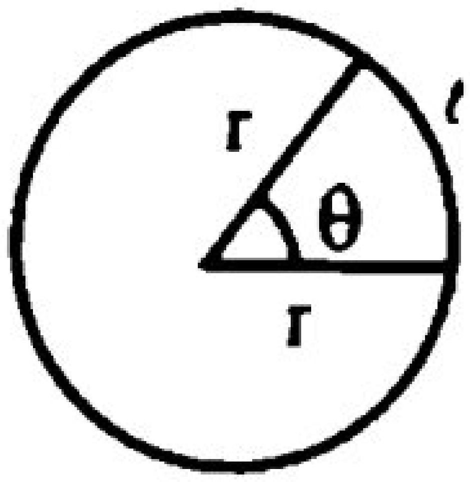


Area of a circular sector

Consider a circle of radius r units and an arc of length units, subtending an angle θ at O.

Area of the circle $= \pi r^2$

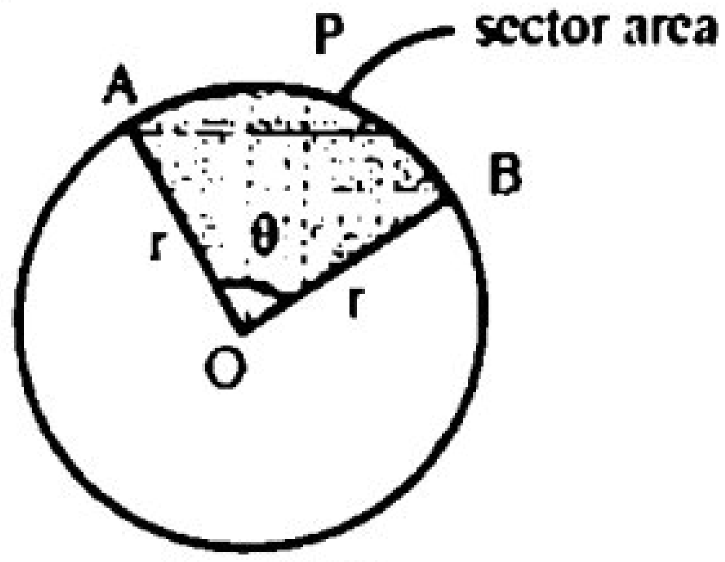


Fig. 7.2.2

Angle of the circle $= \pi r^2$

Angle of the circle $= \pi r^2$

Angle of the sector = 0 radian

Then by elementary geometry we can use the proportion,

$$\frac{\text{area of sector AOBP}}{\text{area of circle}} = \frac{\text{angle of the sector}}{\text{angle of the circle}}$$

or $$\frac{\text{area of sector AOBP}}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\Rightarrow \text{ area of sector AOBP } = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2\theta \text{ angle of the sector AOBP } = \frac{1}{2} r^2\theta$$

SOLVED EXERCISE 7.2

1. Find θ, when:

(i) l = 4.5m, r = 3.5m

Solution

We know that

$l = r\theta \qquad \Rightarrow 0 = \frac{l}{r}$

$0 = \frac{2}{3.5} \qquad \Rightarrow 0 = 0.57 \text{ radians}$

(ii) $l = 4.5m$, $r = 2.5m$

Solution

We know that

$l = r\theta \qquad \Rightarrow \theta = \frac{l}{r}$

$\theta = \frac{4.5}{2.5} \qquad \Rightarrow \theta = 1.8 \text{ radians}$

2. Find l, when

(i) $\theta = 180°$, $r = 4.9$ cm

Solution

$\theta = 180° = 180° \times 1° = 180° \times \frac{\pi}{180°} = 3.1415$ radian

We know that

$l = r\theta \qquad \Rightarrow \theta = (4.9)(3.1415)$

$\theta = 15.4cm$

(ii) $\theta = 60° 30'$, $r = 15mm$

Solution

$\theta = 60°34' = 60° + \left(\frac{34}{60}\right)^{\circ} = 60° + 0.57 = 60.57°$

$= 60.57 \times 1° = 67.57° \times \frac{\pi}{180} = 1.057$ radians

We know that

$l = r\theta \qquad \Rightarrow \qquad \theta = (15)(1.057)$

$\theta = 15.9mm$

3. Find r, when:

(i) $l = 4cm$, $\theta = \frac{1}{4}$ radian

Solution

We know that

$l = r\theta \qquad \Rightarrow \qquad r = \frac{r}{\theta}$

$r = \frac{4}{1/4} \Rightarrow \qquad r = 4 \times \frac{4}{1} = 16cm$

(ii) $l = 52cm$, $\theta = 45°$

Solution

$$\theta = 45^\circ = 45 \times 1^\circ = 45 \times \frac{\pi}{180^\circ} = \frac{\pi}{4}$$
$$= 0.7854 \text{ radians}$$

We know that

$$l = r\theta \quad \Rightarrow \quad r = \frac{l}{\theta}$$

$$r = \frac{52}{0.7854} \quad \Rightarrow \quad r = 66.21\text{cm}$$

4. In a circle of radius 12m, find the length of an arc which subtands a central angle θ = 1.5 radian.

Solution

$r = 12$ cm, $\theta = 1.5$ radians, $l = ?$

We know that

$$l = r\theta \quad \Rightarrow \quad l = (12)(1.5) \Rightarrow l = 18 \text{ m}$$

5. In a circle of radius 10m, find the distance travelled by a point moving on this circle if the point makes 3.5 revolution. (3.5 revolution = 7π).

Solution

$r = 10$m, $\theta = 3.5$ revolution $= 7\pi = 7 \times 3.1415 = 22$ rad.

We know that

$$l = r\theta \quad \Rightarrow \quad l = (10)(22) \Rightarrow l = 220 \text{ m}$$

6. What is the circular measure of the angle between the hands of the watch at 3 o'clock?

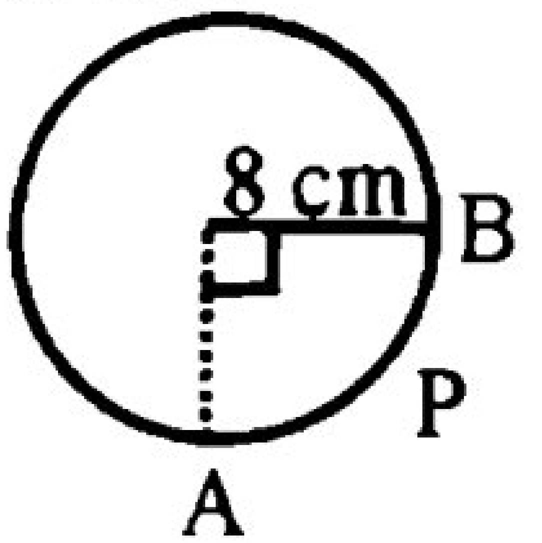
Solution

The angle between the hands of a watch at 3o' clock is

$$= \left(\frac{360}{12} \times 3\right)^\circ = (30 \times 3)^\circ = 90^\circ$$

$$= 90 \times 1^\circ = 90 \times \frac{\pi}{180} = \frac{\pi}{2} \text{ radians}$$

7. What is the length of the arc APB?



Solution

Arc $\widehat{APB} = l = ?$; $r = 8$ cm

$$\theta = 90^\circ = 90 \times 1^\circ = 90 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{2} \text{ radians} = 1.57 \text{ radians}$$

We know that

$l = r\,\theta$ $\Rightarrow$ $l = (8)(1.57) = 12..57$ cm

8. In a circle of radius 12cm, how long an arc subtends a central angle of 84°.

Solution

$$r = 12 \text{ cm}, l = ?$$

$$\theta = 84^\circ = 84 \times 1^\circ = 84 \times \frac{\pi}{180} \text{ radians}$$

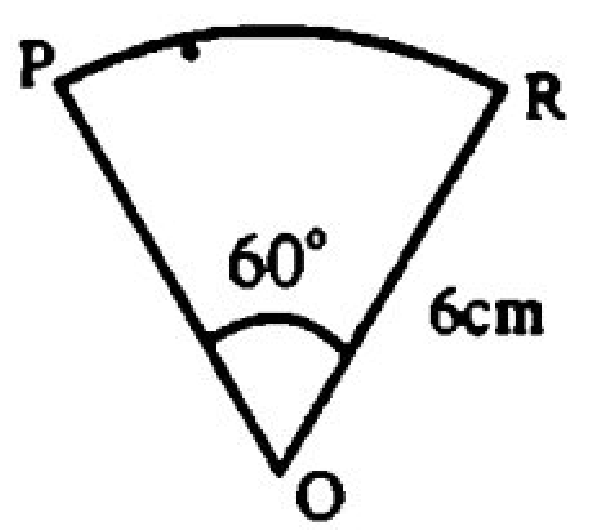
$$= 1.4661 \text{ radians}$$

We know that

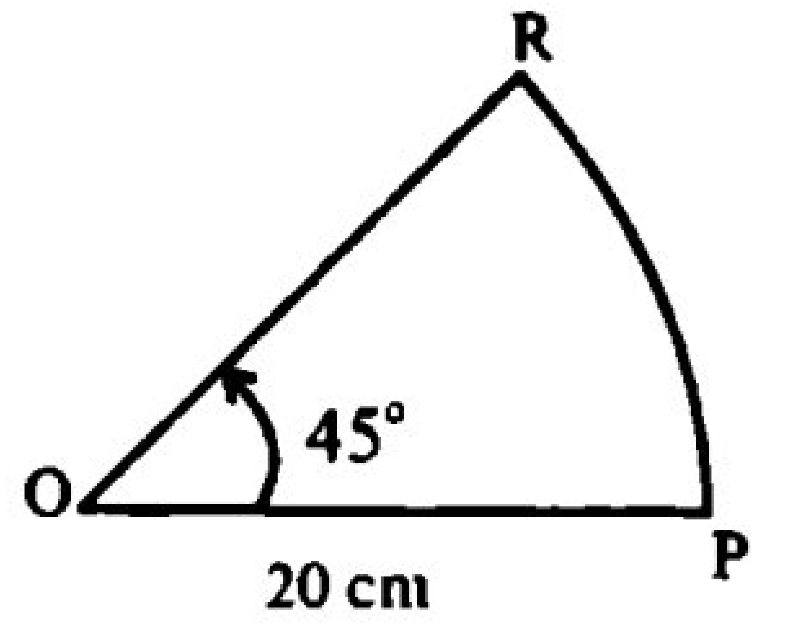
$l = r\theta$ $\Rightarrow$ $l = (12)(1.4661) = 17.6$ cm

9. Find the area of the sectors OPR.

(a)



(b)



Solution

(a) Here $r = 6\text{cm}, \theta = 60^\circ = 60 \times 1^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ rad.

We know that

Area of the sector $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right)$$

$$=\frac{1}{2}(36)\left(\frac{\pi}{3}\right)$$
$$=6\pi$$
$$=6\times\frac{22}{7}$$
$$=18.86\,cm^2$$

(b) Here $r = 20cm$, $\theta = 45° = 45 \times 1° = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad.

We know that

Area of the sector $=\frac{1}{2}r\theta$

$$=\frac{1}{2}(20)^2\left(\frac{\pi}{4}\right)$$
$$=\frac{1}{2}(400)\left(\frac{\pi}{4}\right)$$
$$=50\pi$$
$$=50\times\frac{22}{7}$$
$$=157.14\,cm^2$$

10. Find area of the sector inside a central angle of 20° in a circle of radius 7m.

Solution

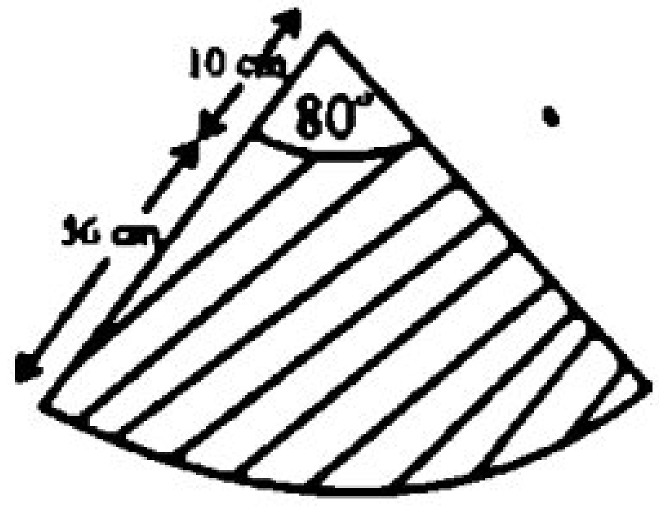
Area = ?, $r = 7m$, $\theta = 20° = 20 \times 1° = 20 \times \frac{\pi}{180}$ rad.

We know that

Area of the sector $=\frac{1}{2}r^2\theta$

$$=\frac{1}{2}(49)\left(\frac{\pi}{9}\right)$$
$$=\frac{49\pi}{18}m^2$$
$$=\frac{49\times22}{18\times7}$$
$$=\frac{77}{9}$$
$$=8.55\,cm^2$$

11. Sehar is making a skirt. Each panel of this skirt is of the shape shown shaded in the diagram. How much material (cloth) is required for each panel?



Solution

$r = 56\text{ cm} + 10\text{ cm} = 66\text{ cm}$

$\theta = 80^\circ = 80 \times 1^\circ = 80 \times \frac{\pi}{180} = \frac{4}{9}\pi = \frac{4}{9} \times \frac{22}{7}\text{ rad.} = \frac{88}{63}\text{ radians}$

We know that

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(66)^2\left(\frac{88}{63}\right)$$

$$= \frac{66 \times 66 \times 88}{2 \times 63}$$

$$= 3\text{cm}^2$$

12. Find the area of the sector with central angle of $\frac{\pi}{5}$ radian in a circle of radius 10cm.

Solution

$\theta = \frac{\pi}{5}\text{ rad.}$ $\quad r = 10\text{ cm},$ $\quad \text{Area} = ?$

$$\text{Area of the sector} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(10)^2\left(\frac{\pi}{5}\right)$$

$$= \frac{1}{2}(100)\left(\frac{\pi}{5}\right)$$

$$= 10\pi \quad \text{cm}^2$$

$$= 10 \times \frac{22}{7}$$

$$= 31.43 \quad \text{cm}^2$$

13. The area of the sector with a central angle θ in a circle of radius 2m is 10 square meter. Find θ in radians.

Solution

$r = 2m$, Area $= 10m^2$, $\theta = ?$

We know that

Area of Sector $= \frac{1}{2} r^2\theta$

$$10 = \frac{1}{2}(2)^2\theta$$

$$10 = 2\theta$$

$$\theta = \frac{10}{2}$$

$$\theta = 5 \text{ radians}$$

Trigonometric Rations:

Coterminal Angle

Two or more than two angles with the same initial and terminal sides are called coterminal angles.

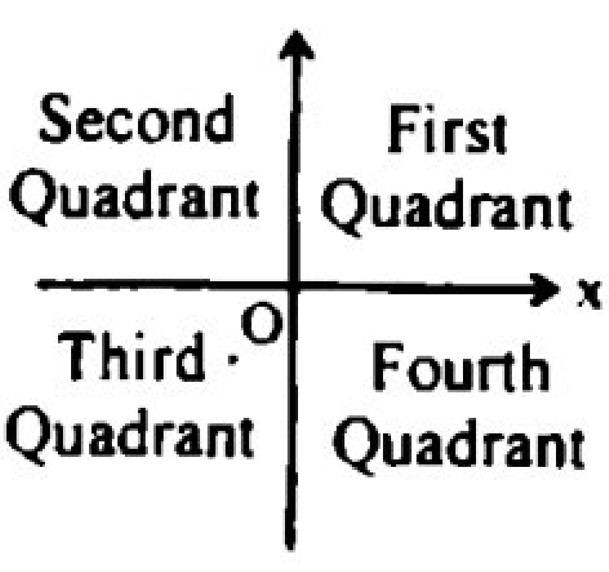
It means that terminal side comes to its original position after every revolution of 2π radian in anti clockwise or clockwise direction. In general if θ is in degrees, then $360^0k + \theta$, where $k \in Z$, is an angle conterminal with θ. If angle θ is in radian measure, then $2k\pi + \theta$ where $k \in Z$, is an angle ceterminal with θ.

Thus, the general angle $\theta = 2(k)\pi + \theta$, where $k \in Z$.

The Quadrants and Quadrantal Angles:

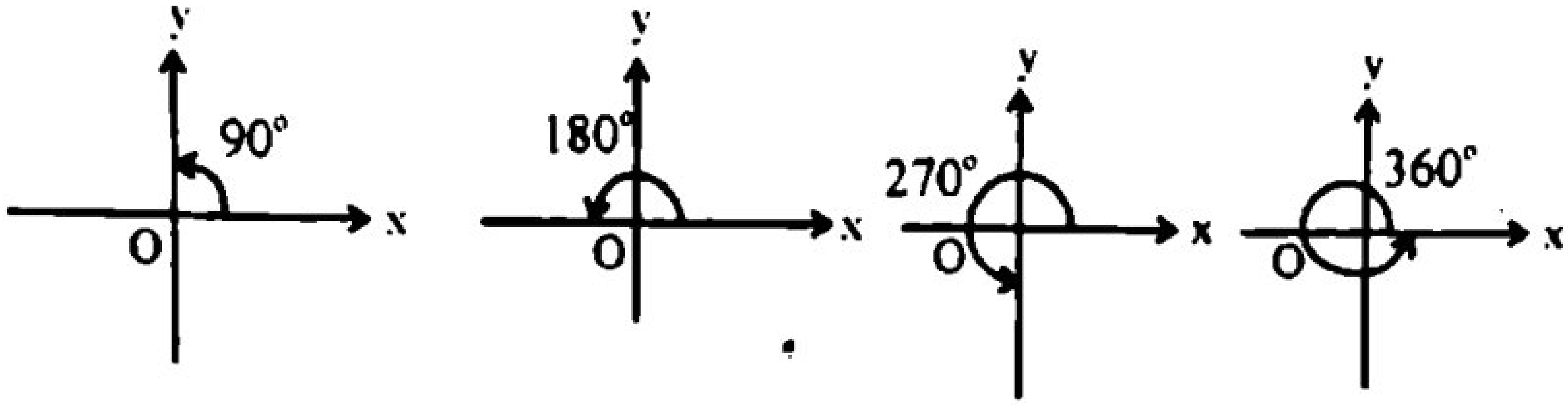
The x – axis and y-axis divides the plane in four regions, called quadrants, when they intersect each other at right angle. The point of intersection is called origin and is denoted by O.

Angles between 0° arid 90° are in the first quadrant.
Angles between 90° and 180° are in the second quadrant.
Angles between 180° and 270° are in the third quadrant.
Angles between 270° to 360° are in the fourth quadrant.
An angle in standard position is said to lie in a quadrant if its terminal side lies in that quadrant. Angles α, β, γ and δ lie in I, II, III and IV quadrant respectively.



Quadrantal Angles:

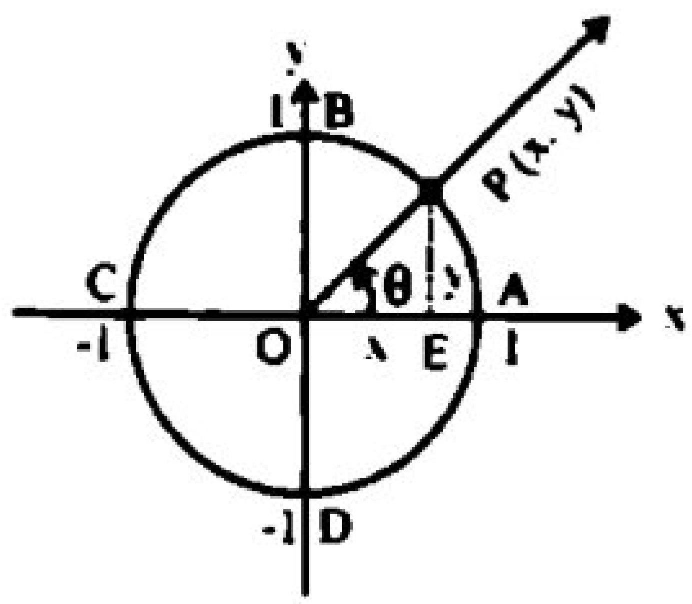
If the terminal side of an angle in standard position falls on x-axis or y-axis, then it is called a quadrantal angle i.e., 90°, 180°, 270° and 360° are quadrantal angles. The quadrantal angles are shown as below:



Trigonometric ratios and their reciprocals with the help of a unit circle:

There are six fundamental trigonometric ratios called sine, cosine, tangent, cotangent, secant and cosecant. To define these; functions we use circular approach which involves the unit circle.

Let θ be a real number, which represents the radian measure of an angle in standard position. Let P (x, y) be any point on the unit circle lying on terminal side of θ as shown in the figure.



We define sine of θ, written as sinθ and cosine of θ written as cosθ as:

i.e., cosθ and sinθ are the x-coordinate and y-coordinate of the point P on the unit circle. The equations x = cos θ and y = sin θ are called circular or trigonometric functions.

The remaining trigonometric functions tangent, cotangent, secant and cosecant will be denoted by tan θ, cotθ, secθ and cosec θ for any real angle θ.

Reciprocal Identities

$\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$	or	$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	or	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	or	$\cot\theta = \dfrac{1}{\tan\theta}$

The values of trigonometric ratio for 45°, 30°, 60°:

Consider a right triangle ABC with m ∠ C = 90°. The sides opposite to the vertices A, B and C are denoted by a, b and c respectively.

Case I

When m ∠ A = 45°, where $45° = \dfrac{\pi}{4}$ radian. Since the sum of angles in a triangle is 180°, so m ∠ B = 45°.

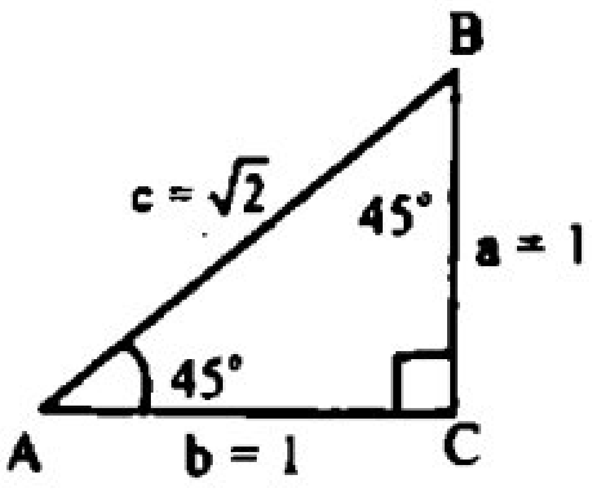


Fig. 7.3 5

. As values of trigonometric functions depends on the size of the angle only and not on the size of triangle. For convenience, we take a = b = 1. In this ease the triangle is isosceles right triangle.

By Pythagorean theorem, we have

$$c^2 = a^2 + b^2 \Rightarrow c^2 = (1)^2 + (1)^2 = 2$$

$$c^2 = 2 \quad \Rightarrow c = \sqrt{2}$$

From this triangle we have

$$\sin 45° = \sin\frac{\pi}{4} = \frac{a}{c} = \frac{1}{\sqrt{2}}; \qquad \operatorname{cosec} 45° = \frac{1}{\sin 45°} = \sqrt{2}$$

$$\cos 45° = \cos\frac{\pi}{4} = \frac{b}{c} = \frac{1}{\sqrt{2}}; \qquad \sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$$

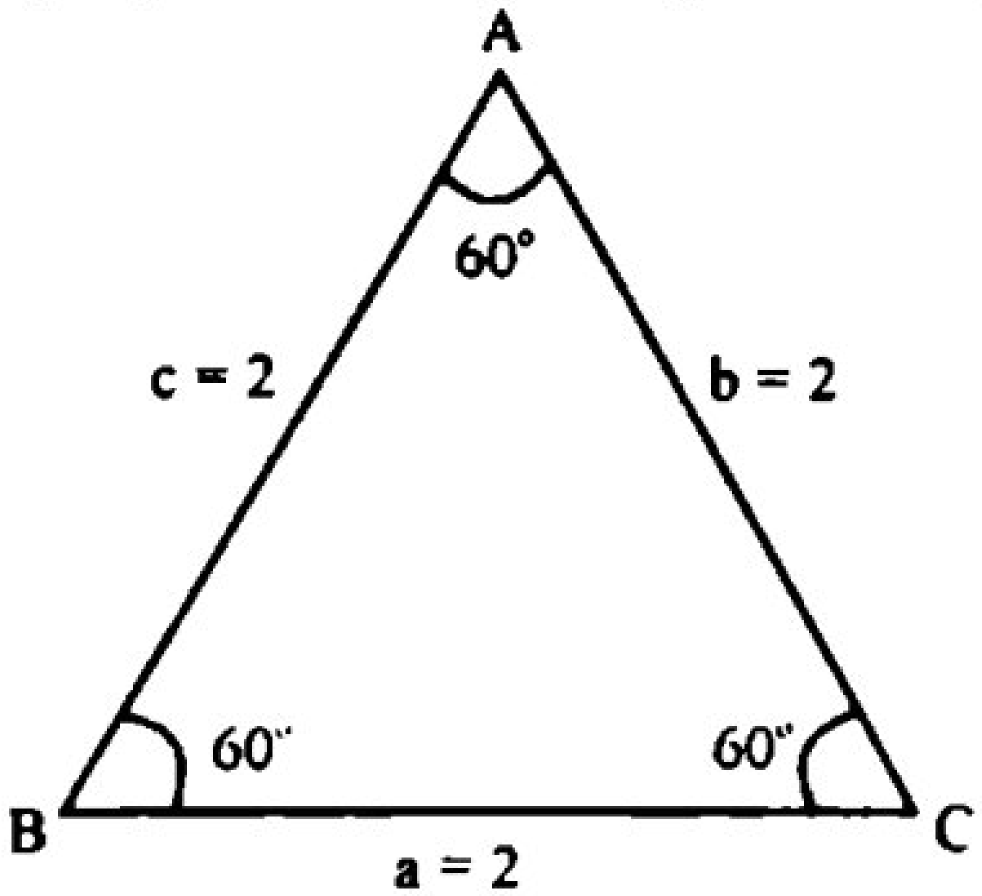
$$\tan 45° = \tan\frac{\pi}{4} = \frac{a}{b} = \frac{1}{1} = 1; \qquad \cot 45° = \frac{1}{\tan 45°} = 1$$

Case II

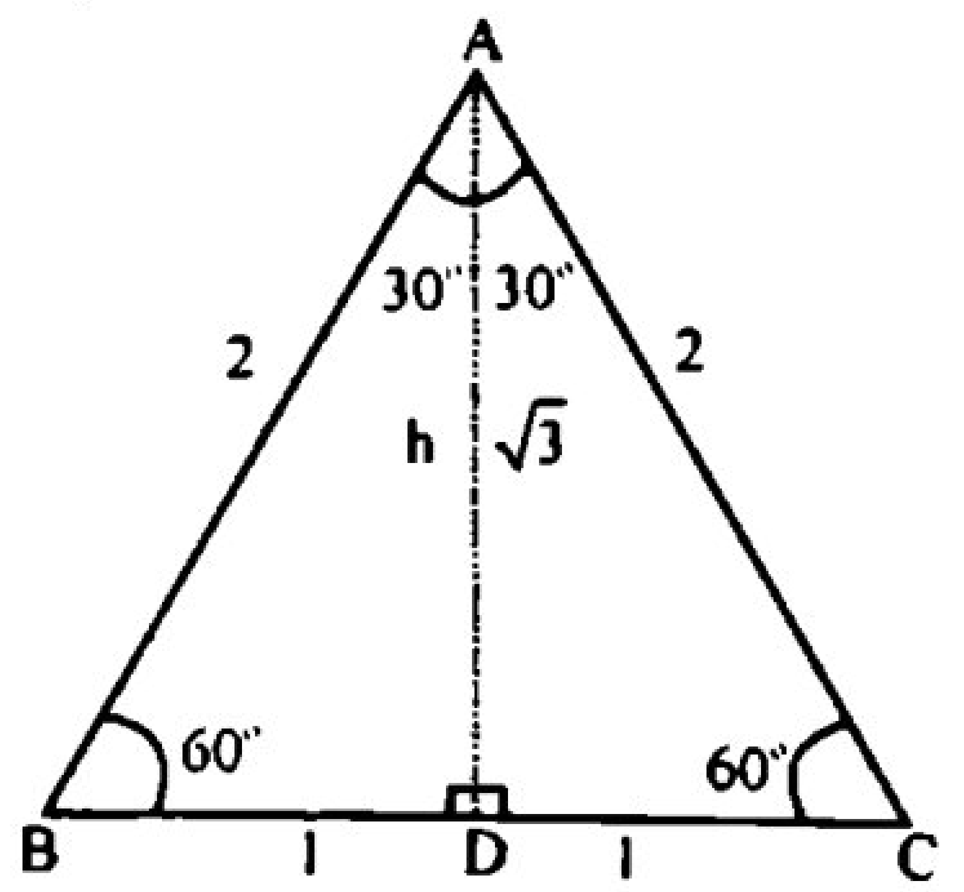
When $m\angle A = 30°$ or $m\angle A = 60°$

Consider an equilateral triangle with sides a = b = c = 2 for convenience. Since the angles in an equilateral triangle are equal and their sum is 180°, each angle has measure 60°.

Bisecting an angle in the triangle, we obtain two right triangles with 30° and 60° angles. The height |AD| of these triangles may be found by Pythagorean theorem, i.e.,



(i)



(ii)

$$(AD)^2 = (AB)^2 - (AD)^2 \Rightarrow \qquad (AD)^2 = (AB)^2 - (BD)^2$$

$$h^2 = (2)^2 - (1)^2 = 3$$

$$\Rightarrow \quad h = \sqrt{3}$$

Using triangle ADB with m ∠ A = 30°, we have

$$\sin 30° = \sin\frac{\pi}{6} = \frac{BD}{AB} = \frac{1}{2} \qquad \operatorname{cosec}30° = \frac{1}{\sin 30°} = 2$$

$$\cos 30° = \cos\frac{\pi}{6} = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \sec 30° = \frac{1}{\cos 30°} = 2$$

$$\tan 30° = \tan\frac{\pi}{6} = \frac{BD}{AD} = \frac{1}{\sqrt{3}} \qquad \cot 30° = \frac{1}{\sin 30°} = \sqrt{3}$$

Now using triangle ABD with m ∠ B = 60°.

$$\sin 60° = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \operatorname{cosec}60° = \frac{1}{\sin 60°} = \frac{2}{\sqrt{3}}$$

$$\cos 60° = \frac{BD}{AB} = \frac{1}{2} \qquad \sec 60° = \frac{1}{\cos 60°} = 2$$

$$\tan 60° = \frac{AD}{BD} = \frac{\sqrt{3}}{1} \qquad \cot 60° = \frac{1}{\tan 60°} = \frac{1}{\sqrt{3}}$$

Signs of trigonometric ratios in different quadrants:

In case of trigonometric ratios like sinθ cosθ and tanθ if θ is not a quadrantal angle, then θ will lie in a particular quadrant. Since $r = \sqrt{x^2 + y^2}$ is always + ve, the signs of ratios can be found if the quadrant of 6 is known.

(i) If θ lies in first quadrant then a point P (x, y) on its terminal side has x and y co-ordinate positive.
Therefore, all trigonometric functions are positive in quadrant I

(ii) If θ lies in 2^{nd} quadrant, then point P (x, y) on its terminal side has negative Jc-coordinate and positives-coordinate.

$$\therefore \sin\theta = \frac{y}{r} \text{ is + ve or} > 0 \ \cos\theta = \frac{x}{r} \text{ is – ve or} < 0 \text{ and } \tan = \frac{y}{x} \text{ is – ve or} < 0$$

(iii) When θ lies in third quadrant, then a point P (x, y) on its terminal side has negative x-coordinale and negatives-coordinate.

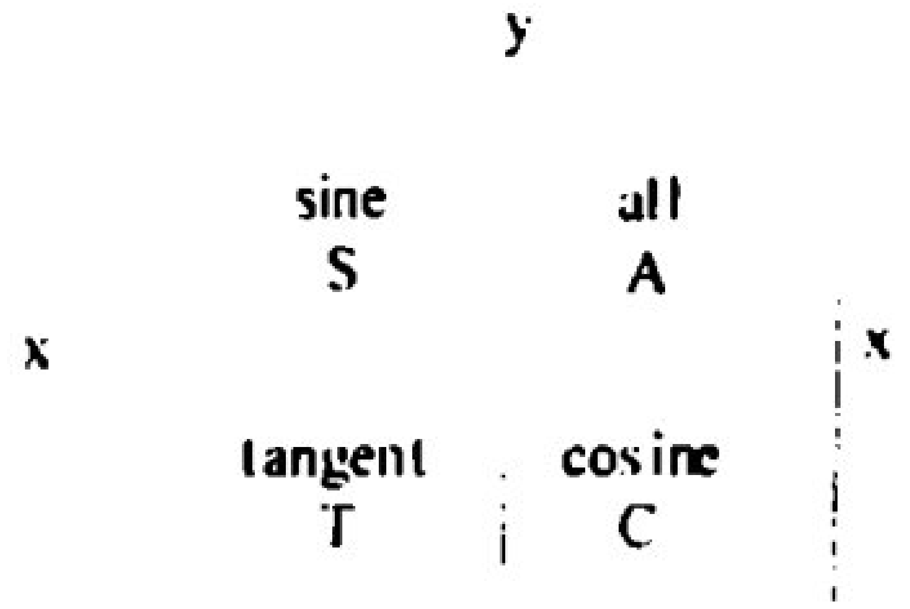
$$\therefore \sin\theta = \frac{y}{r} \text{ is + ve or} < 0 \ \cos\theta = \frac{x}{r} \text{ is – ve or} < 0 \text{ at } \tan\theta = \frac{y}{x} \text{ is – ve or} > 0$$

(iv) When θ lies in fourth quadrant, then the point P (x, y) on the terminal side of θ has positive x-coordinate and negatives-coordinate.

$$\therefore \sin 0 = \frac{y}{r} \text{ is – ve or} < 0 \ \cos\theta = \frac{x}{r} \text{ is + ve or} > 0 \text{ and } \tan\theta = \frac{y}{x} \text{ is – ve or} < 0$$

The signs of all trigonometric functions, are summarized as below.

$\sin\theta > 0$	$\sin\theta > 0$
$\operatorname{cosec}\theta > 0$	$\operatorname{cosec}\theta > 0$
$\cos\theta < 0$	$\cos\theta > 0$
$\sec\theta < 0$	$\sec\theta > 0$
$\tan\theta < 0$	$\tan\theta > 0$
$\cot\theta < 0$	$\cot\theta > 0$
$\tan\theta > 0$	$\cos\theta > 0$
$\cot\theta > 0$	$\sec\theta > 0$
$\sin\theta < 0$	$\sin\theta < 0$
$\operatorname{cosec}\theta < 0$	$\operatorname{cosec}\theta < 0$
$\cos\theta < 0$	$\tan\theta < 0$
$\sec\theta < 0$	$\cot\theta < 0$



Calculate the values of trigonometric ratios for 0°, 90°, 180°, 270°, 360°:

We have discussed quadrantal angles in section 7.3.2. An angle 6 is called a quadrantal angle if its terminal side lies on the x-axis or on the y-axis.

Case I

When $\theta = 0^\circ$

The point (1, 0) lies on the terminal side of angle θ°. We may consider the point on the unit circle on the terminal side of the angle.

$$P(1,0) \Rightarrow x = 1 \text{ and } y = 0 \quad \text{so } r = \sqrt{x^2 + y^2} = \sqrt{1+0} = 1$$

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 1, \quad \operatorname{cosec} 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0^\circ} = \infty(\text{undefined})$$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1, \quad \sec 0^\circ = \frac{1}{\cos 0^\circ} = 1$$

$$\tan 0^\circ = \frac{y}{r} = \frac{0}{1} = 1, \quad \cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0^\circ} = \infty(\text{undefined})$$

Case II When θ = 90°

The point P (0, 1) lies on the terminal side of angle 90°

Here $x = 0$ and $y = 1 \quad \Rightarrow \quad r = \sqrt{0^2 + (1)^2} = 1$

$$\sin 90^\circ = \frac{y}{r} = \frac{1}{1} = 1$$

i.e., $\sin 90^\circ = 1$ and $\operatorname{cosec} 90^\circ = \frac{r}{y} = 1$

Using reciprocal identities, we have

$$\tan 90^\circ = \frac{y}{x} = \frac{1}{0} = \infty(\text{undefined}), \cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

Case III When $\theta° = 180°$

The point P $(-1, 0)$ lies on x'-axis or on terminal side of angle 180°

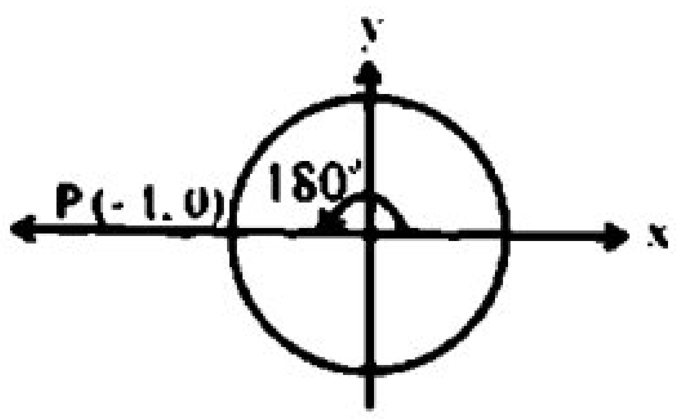
Here $x = -1$ and $y = 0$

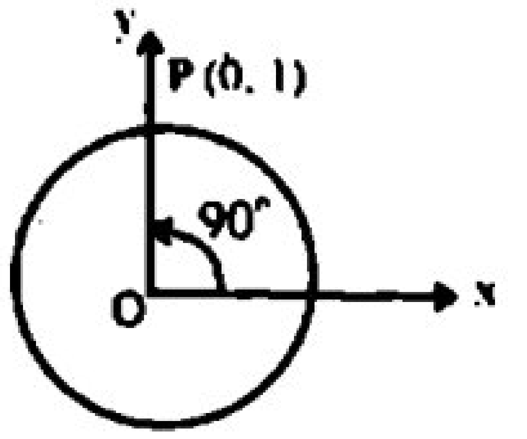
$$\Rightarrow \quad = \sqrt{x^2 + y^2} = 1$$

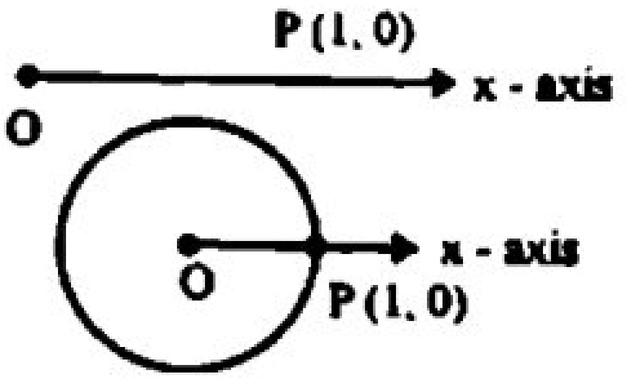
$$\therefore \sin 180° = \frac{y}{r} = \frac{0}{1} = 0 \quad ; \quad \operatorname{cosec} 180° = \frac{r}{y} = \frac{1}{0} = \quad \text{(undefined)}$$

$$\cos 180° = \frac{x}{r} = \frac{-1}{1} = -1 \quad ; \quad \sec 180° = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0 \quad ; \quad \cot 180° = \frac{x}{y} = \frac{-1}{0} = \infty \quad \text{(undefined)}$$







Case IV

When $\theta = 270°$ and the point P $(0, -1)$ lies on y-axis or on the terminal side of angle 270°.

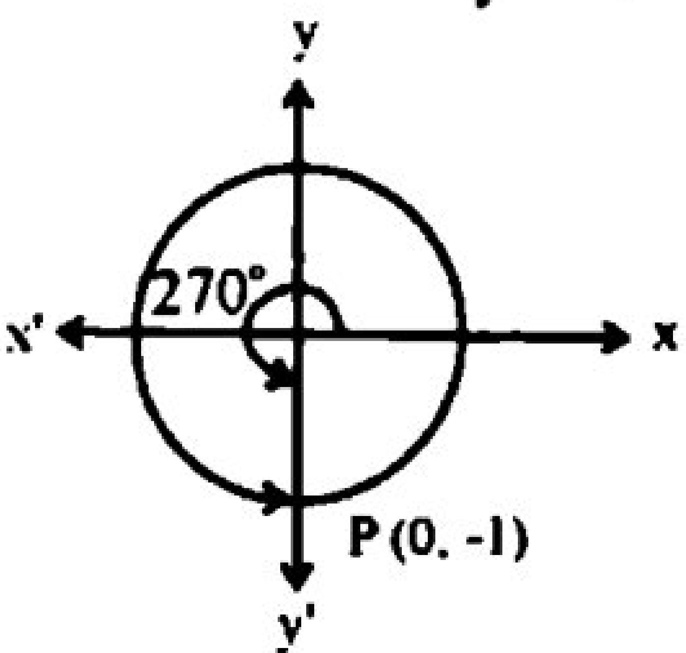
The point P $(0, -1)$ shows that $x = 0$ and $y = -1$

So $\quad r = \sqrt{(0)^2 + (-1)^2} = 1$

$$\sin 270° = \frac{y}{r} = \frac{-1}{1} = -1 \quad ; \quad \operatorname{cosec} 270° = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos 270° = \frac{x}{r} = \frac{0}{1} = 0 \quad ; \quad \sec 270° = \frac{r}{x} = \frac{1}{0} = \infty$$

$$\tan 270° = \frac{y}{x} = \frac{-1}{0} = -\infty \quad ; \quad \cot 270° = \frac{x}{y} = \frac{0}{-1} = 0$$



Case V

When $\theta° = 360°$

Now the point P $(1, 0)$ lies once again on x – axis

We know that $2k\pi + \theta - \theta$ where $k \in Z$.
Now $\theta = 360° = 0° + (360°)\,1 = 0°$ where $k = 1$

So $\sin 360° = \sin 0° = 0$; $\operatorname{cosec} 360° = \dfrac{1}{\sin 360°} = \dfrac{1}{\sin 0°} = \dfrac{1}{0} = \infty$ (undefined)

$\cos 360^0 = \cos b° = 1$; $\sec 360° = \dfrac{1}{\cos 0°} = \dfrac{1}{1} = 1,$

$\tan 360^0 = \tan 0^0 = 0$; $\cot 360° = \dfrac{1}{\tan 0°} = \dfrac{1}{0} = \infty$ (undefined)

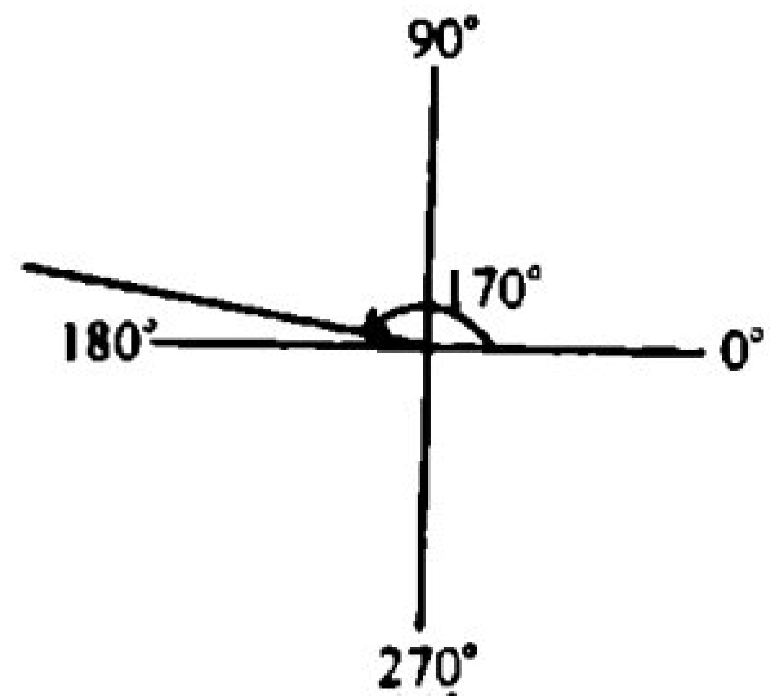
SOLVED EXERCISE 7.3

1. Locate each of the following angles in standard position using a protractor or fair free hand guess. Also find a positive and a negative angle conterminal with each given angle.

(i) 170°

Solution

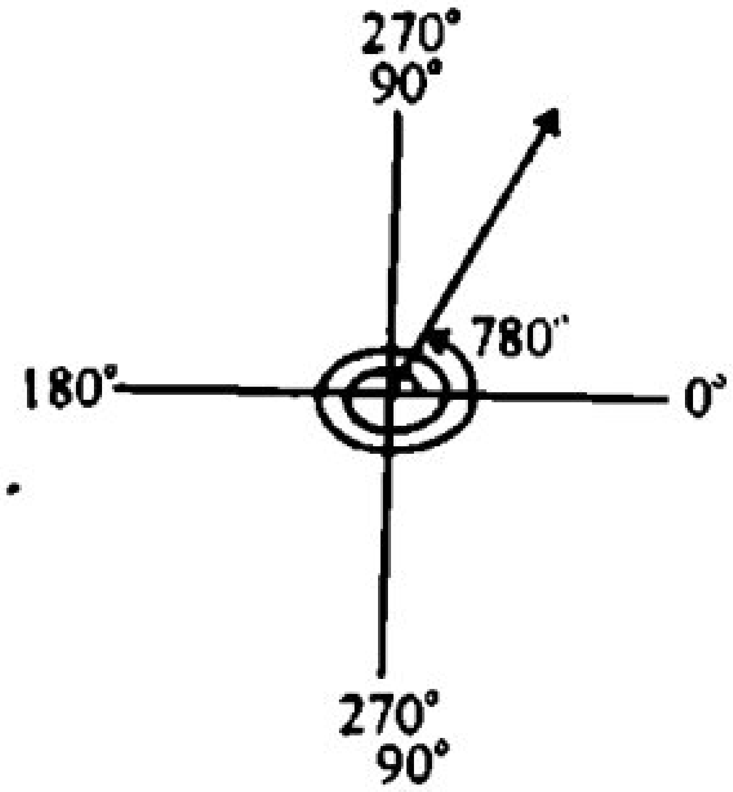
Positive conterminal angle
$= 360° + 170°$
$= 530°$
Negative Coterminal angle
$= 360 - 170°$
$= 190°$



(ii) 780°

Solution

Positive conterminal angle
$= 780° - 360° - 360°$
$= 780° - 720°$
$= 60°$



(iii) −100°

Solution

Positive conterminal angle
$= 360° - 100°$
$= 260°$
Negative Conterminal angle
$= -360° - 100°$
$= -460°$