

SOLVED EXERCISE 7.4

In Problems 1– 6, simplify each expression to a single trigonometric function.

1. $\frac{\sin^2 x}{\cos^2 x}$

Solution

$$\begin{aligned}\frac{\sin^2 x}{\cos^2 x} &= \left(\frac{\sin x}{\cos x} \right)^2 \\ &= (\tan)^2 \\ &= \tan^2 x\end{aligned}\quad \because \frac{\sin x}{\cos x} = \tan x$$

2. $\tan x \sin x \sec x$

Solution

$$\begin{aligned}\tan x \sin x \sec x &= \frac{\sin x}{\cos x} \cdot \sin x \cdot \frac{1}{\cos x} \\ &= \frac{\sin x \cdot \sin x}{\cos x \cdot \cos x} \\ &= \frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin x}{\cos x} \right)^2 \\ &= (\tan x)^2 \\ &= \tan^2 x\end{aligned}\quad \because \frac{\sin x}{\cos x} = \tan x$$

3. $\frac{\tan x}{\cos x}$

Solution

$$\begin{aligned}\frac{\tan x}{\cos x} &= \frac{\tan x}{\cos x} = \frac{\sin x / \cos x}{1 / \cos x} \\ &= \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} \\ &= \sin x\end{aligned}$$

4. $1 - \cos^2 x$

Solution

$$\begin{aligned}
 1 - \cos^2 x &= 1 - (1 - \sin^2 x) \\
 &= 1 - 1 + \sin^2 x \\
 &= \sin^2 x
 \end{aligned}$$

$$\therefore \cos^2 x = 1 - \sin^2 x$$

5. $\sec^2 x - 1$

Solution

$$\begin{aligned}
 1 - \cos^2 x &= \frac{1}{\cos^2 x} - 1 \\
 &= \frac{1 - \cos^2}{\cos^2 x} \\
 &= \frac{\sin^2 x}{\cos^2 x} \\
 &= \left(\frac{\sin x}{\cos x} \right)^2 \\
 &= (\tan x)^2 \\
 &= \tan^2 x
 \end{aligned}$$

$$\therefore \sec x = \frac{1}{\cos x}$$

$$\therefore 1 - \cos^2 x = \sin^2 x$$

6. $\sec^2 x \cot^2 x$

In problems 7 – 24, verify the identities.

Solution

$$\begin{aligned}
 \sin^2 x \cot^2 x &= \sin^2 x \cdot \frac{\cos^2 x}{\sin^2 x} \\
 &= \cos^2 x
 \end{aligned}$$

$$\therefore \cot x = \frac{\cos x}{\sin x}$$

In problem 7 – 24, verify the identities.

7. $(1 - \sin \theta)(1 + \sin \theta) = \cos^2 \theta$

Solution

$$\begin{aligned}
 \text{L.H.S.} &= (1 - \sin \theta)(1 + \sin \theta) \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved

8. $\frac{\sin \theta + \cos \theta}{\cos \theta} = 1 + \tan \theta$

Solution

$$\text{L.H.S.} = \frac{\sin \theta + \cos \theta}{\cos \theta}$$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\cos \theta}$$

$$= \tan \theta + 1$$

$$= 1 + \tan \theta$$

$$= \text{R.H.S.}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = 1$$

Hence proved.

$$9. (\tan \theta + \cot \theta) \tan \theta = \sec^2 \theta$$

Solution

$$\text{L.H.S.} = (\tan \theta + \cot \theta) \tan \theta$$

$$= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)$$

$$= \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \right) \cdot \frac{\sin \theta}{\cos \theta}$$

$$= \left(\frac{1}{\sin \theta \cos \theta} \right) \frac{\sin \theta}{\cos \theta}$$

$$\therefore \sin^2 \theta + \cos^2 \theta = 1$$

$$= \frac{1}{\cos^2 \theta}$$

$$= \sec^2 \theta$$

$$= \text{R.H.S.}$$

Hence proved.

$$10. (\cot \theta + \operatorname{cosec} \theta)(\tan \theta - \sin \theta) = \sec \theta - \cos \theta$$

Solution

$$\text{L.H.S.} = (\cot \theta + \operatorname{cosec} \theta)(\tan \theta - \sin \theta)$$

$$\begin{aligned}
&= \left(\frac{\cot \theta}{\sin \theta} + \frac{1}{\sin \theta} \right) \left(\frac{\sin \theta}{\cos \theta} - \sin \theta \right) \\
&= \left(\frac{\cos \theta + 1}{\sin \theta} \right) \left(\frac{\sin \theta - \sin \theta \cos \theta}{\cos \theta} \right) \\
&= \left(\frac{1 + \cos \theta}{\sin \theta} \right) \left[\frac{\sin \theta (1 - \cos \theta)}{\cos \theta} \right] \\
&= \frac{1 + \cos \theta (1 - \cos \theta)}{\cos \theta} \\
&= \frac{1 - \cos^2 \theta}{\cos \theta} \\
&= \frac{1}{\cos \theta} - \frac{\cos^2 \theta}{\cos \theta} \\
&= \frac{1}{\cos \theta} - \cos \theta \\
&= \sec \theta - \cos \theta \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

$$11. \frac{\sin \theta + \cos \theta}{\tan^2 \theta - 1} = \frac{\cos^3 \theta}{\sin \theta - \cos \theta}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\sin \theta + \cos \theta}{\tan^2 \theta - 1} \\
&= \frac{\sin \theta + \cos \theta}{\frac{\sin^2 \theta}{\cos^2 \theta} - 1} \\
&= \frac{\sin \theta + \cos \theta}{\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta}} \\
&= \frac{\cos^2 \theta (\sin \theta + \cos \theta)}{\sin^2 \theta - \cos^2 \theta} \\
&= \frac{\cos^2 \theta (\sin \theta + \cos \theta)}{(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)} \\
&= \frac{\cos^2 \theta}{\sin \theta - \cos \theta} \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

$$12. \frac{\cos^2 \theta}{\sin \theta} + \sin \theta = \operatorname{cosec} \theta$$

Solution:

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos^2 \theta}{\sin \theta} + \sin \theta \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta} \\ &= \frac{1}{\sin \theta} \\ &= \operatorname{cosec} \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$13. \sec \theta - \cos \theta = \tan \theta \sin \theta$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \sec \theta - \cos \theta \\ &= \frac{1}{\cos \theta} - \cos \theta \\ &= \frac{1 - \cos^2 \theta}{\cos \theta} \\ &= \frac{\sin^2 \theta}{\cos \theta} \\ &= \frac{\sin \theta}{\cos \theta} \cdot \sin \theta \\ &= \tan \theta \sin \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$14. \frac{\sin^2 \theta}{\cos \theta} + \cos \theta = \sec \theta$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin^2 \theta}{\cos \theta} + \cos \theta \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} \\ &= \frac{1}{\cos \theta} \\ &= \sec \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$15. \tan \theta + \cot \theta = \sec \theta \operatorname{cosec} \theta$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \tan \theta + \cot \theta \\ &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta} \cdot \frac{1}{\cos \theta} \\ &= \operatorname{cosec} \theta \sec \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$16. (\tan \theta + \cot \theta)(\cos \theta + \sin \theta) = \sec \theta + \operatorname{cosec} \theta$$

Solution

$$\begin{aligned} \text{L.H.S.} &= (\tan \theta + \cot \theta)(\cos \theta + \sin \theta) \\ &= \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)(\cos \theta + \sin \theta) \\ &= \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \right)(\cos \theta + \sin \theta) \\ &= \left(\frac{1}{\sin \theta \cos \theta} \right)(\cos \theta + \sin \theta) \\ &= \frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta} \\ &= \frac{\cos \theta}{\sin \theta \cos \theta} + \frac{\sin \theta}{\sin \theta \cos \theta} \\ &= \frac{1}{\sin \theta} + \frac{1}{\cos \theta} \\ &= \operatorname{cosec} \theta + \sec \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$17. \sin \theta (\tan \theta + \cot \theta) = \sec \theta$$

Solution

$$\text{L.H.S.} = \sin \theta (\tan \theta + \cot \theta)$$

$$\begin{aligned}
 &= \sin \theta \left(\frac{\sin^2 \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) \\
 &= \sin \theta \left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \right) \\
 &= \sin \theta \left(\frac{1}{\sin \theta \cos \theta} \right) \\
 &= \frac{1}{\cos \theta} \\
 &= \sec \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved.

$$18. \frac{1 + \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} = 2 \operatorname{cosec} \theta$$

Solution

$$\begin{aligned}
 \text{L.H.S.} &= \frac{1 + \cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta} \\
 &= \frac{(1 + \cos \theta)^2 + (\sin \theta)^2}{\sin \theta (1 + \cos \theta)} \\
 &= \frac{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}{\sin \theta (1 + \cos \theta)} \\
 &= \frac{1 + 2 \cos \theta + 1}{\sin \theta (1 + \cos \theta)} \\
 &= \frac{2 + 2 \cos \theta}{\sin \theta (1 + \cos \theta)} \\
 &= \frac{2(1 + \cos \theta)}{\sin \theta (1 + \cos \theta)} \\
 &= \frac{2}{\sin \theta} \\
 &= 2 \operatorname{cosec} \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved.

$$19. \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta} = 2 \operatorname{cosec}^2 \theta$$

Solution

$$\text{L.H.S.} = \frac{1}{1 - \cos \theta} + \frac{1}{1 + \cos \theta}$$

$$\begin{aligned}
 &= \frac{1 + \cos \theta + 1 - \cos \theta}{(1 - \cos \theta)(1 + \cos \theta)} \\
 &= \frac{2}{1 - \cos^2 \theta} \\
 &= \frac{2}{\sin^2 \theta} \\
 &= 2 \operatorname{cosec}^2 \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved.

$$20. \frac{1 + \sin \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{1 + \sin \theta} = 4 \tan \theta \sec \theta$$

Solution

$$\begin{aligned}
 \text{L.H.S.} &= \frac{1 + \sin \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{1 + \sin \theta} \\
 &= \frac{(1 + \sin \theta)^2 - (1 - \sin \theta)^2}{(1 - \sin \theta)(1 + \sin \theta)} \\
 &= \frac{(1 + 2\sin \theta + \sin^2 \theta) - (1 - 2\sin \theta + \sin^2 \theta)}{1 - \sin^2 \theta} \\
 &= \frac{1 + 2\sin \theta + \sin^2 \theta - 1 + 2\sin \theta - \sin^2 \theta}{\cos^2 \theta} \\
 &= \frac{4\sin \theta}{\cos^2 \theta} \\
 &= 4 \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} \\
 &= 4 \tan \theta \sec \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved.

$$21. \sin^3 \theta = \sin \theta - \sin \theta \cos^2 \theta$$

Solution

$$\begin{aligned}
 \text{L.H.S.} &= \sin \theta - \sin \theta \cos^2 \theta \\
 &= \sin \theta (1 - \cos^2 \theta) \\
 &= \sin \theta \cdot \sin^2 \theta \\
 &= \sin^3 \theta \\
 &= \text{R.H.S.}
 \end{aligned}$$

Hence proved.

$$22. \cos^4 \theta - \sin^4 \theta = (\cos^2 \theta - \sin^2 \theta)$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \cos^4 \theta - \sin^4 \theta \\ &= (\cos^2 \theta)^2 - (\sin^2 \theta)^2 \\ &= (\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) \\ &= (\cos^2 \theta - \sin^2 \theta)(1) \\ &= \cos^2 \theta - \sin^2 \theta \\ &= \text{R.H.S.} \end{aligned}$$

Hence proved.

$$23. \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} = \frac{\sin \theta}{1 - \cos \theta}$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}} \\ &= \frac{\sqrt{1 + \cos \theta}}{\sqrt{1 - \cos \theta}} \times \frac{\sqrt{1 + \cos \theta}}{\sqrt{1 + \cos \theta}} \\ &= \frac{(\sqrt{1 - \cos \theta})^2}{\sqrt{(1)^2 - (\cos \theta)^2}} \\ &= \frac{1 + \cos \theta}{\sqrt{1 - \cos^2 \theta}} \\ &= \frac{1 + \cos \theta}{\sqrt{\sin^2 \theta}} \\ &= \frac{1 + \cos \theta}{\sin \theta} \times \frac{1 - \cos \theta}{1 - \cos \theta} \\ &= \frac{(1)^2 - (\cos \theta)^2}{\sin \theta (1 - \cos \theta)} \\ &= \frac{1 - \cos^2 \theta}{\sin \theta (1 - \cos \theta)} \\ &= \frac{1 - \cos^2 \theta}{\sin \theta (1 - \cos \theta)} \end{aligned}$$

$$= \frac{\sin^2 \theta}{\sin \theta (1 - \cos \theta)}$$

$$= \frac{\sin \theta}{1 - \cos \theta}$$

$$= \text{R.H.S.}$$

Hence proved.

$$24. \sqrt{\frac{\sec \theta + 1}{\sec \theta - 1}} = \frac{\sec \theta + 1}{\tan \theta}$$

Solution

$$\begin{aligned} \text{L.H.S.} &= \sqrt{\frac{\sec \theta + 1}{\sec \theta - 1}} \\ &= \frac{\sqrt{\sec \theta + 1}}{\sqrt{\sec \theta - 1}} \\ &= \frac{\sqrt{\sec \theta + 1}}{\sqrt{\sec \theta - 1}} \times \frac{\sec \theta + 1}{\sec \theta + 1} \\ &= \frac{(\sqrt{\sec \theta + 1})^2}{\sqrt{\sec^2 \theta - 1}} \\ &= \frac{\sec \theta + 1}{\sqrt{\tan^2 \theta}} \\ &= \frac{\sec \theta + 1}{\tan \theta} \\ &= \text{R.H.S.} \end{aligned}$$

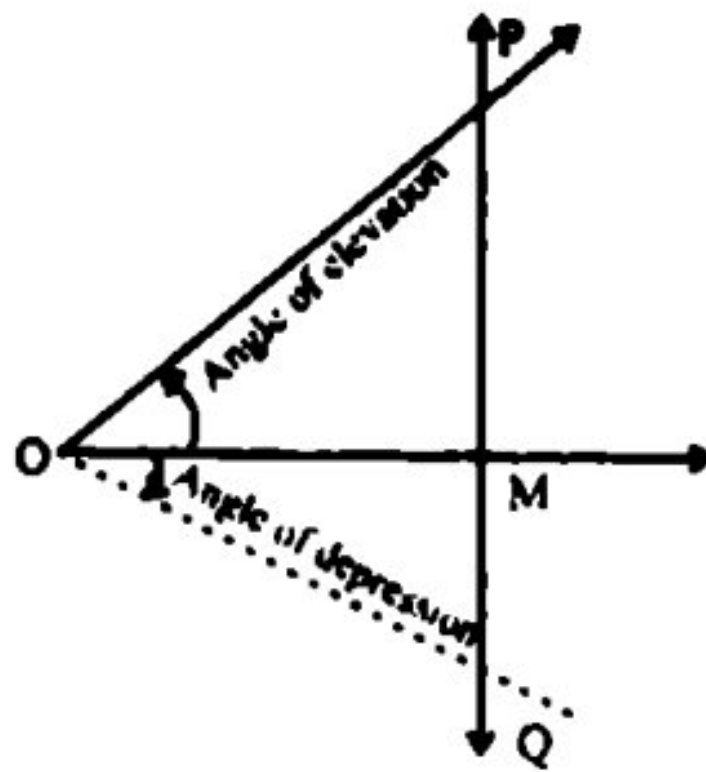
Hence proved.

Angle of Elevation and Angle of Depression:

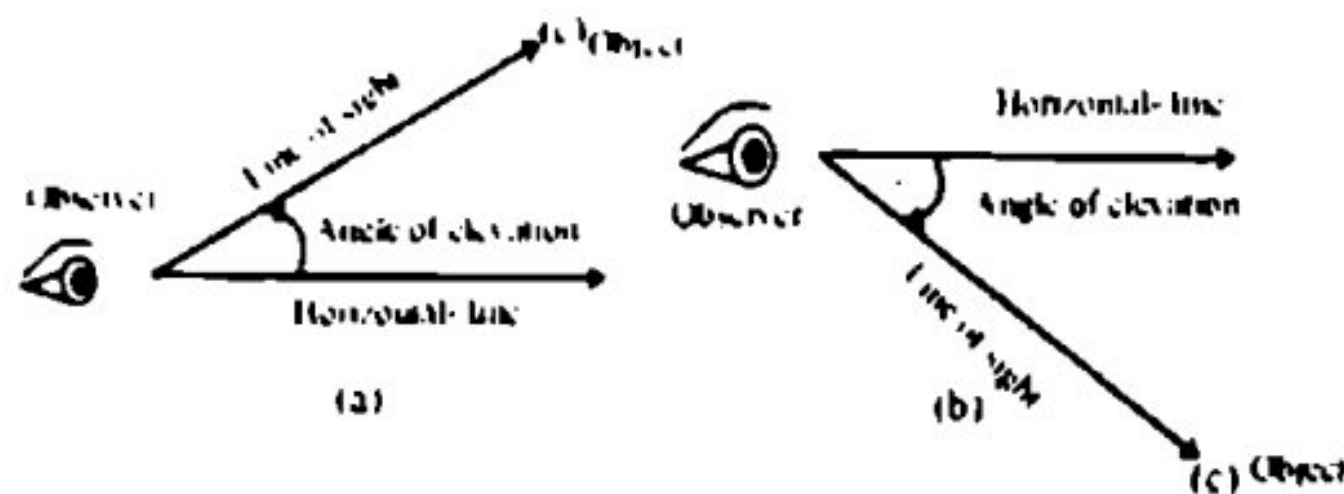
One of the objects of trigonometry is to find the distances between points or the heights of objects, without actually measuring these distances or heights.

Angle of elevation: Suppose O, P and Q are three points, P being at a higher level of O and Q being at lower level than O. Let a horizontal line drawn through O meet in M, the vertical line drawn through P and Q.

The angle MOP is called the angle of elevation of point P as seen from O. For looking at Q below the horizontal line we have to lower our eyes and $\angle MOQ$ is called the angle of depression.



We measure an angle of elevation from a horizontal line up to an object or an angle of depression from a horizontal line down to an object,

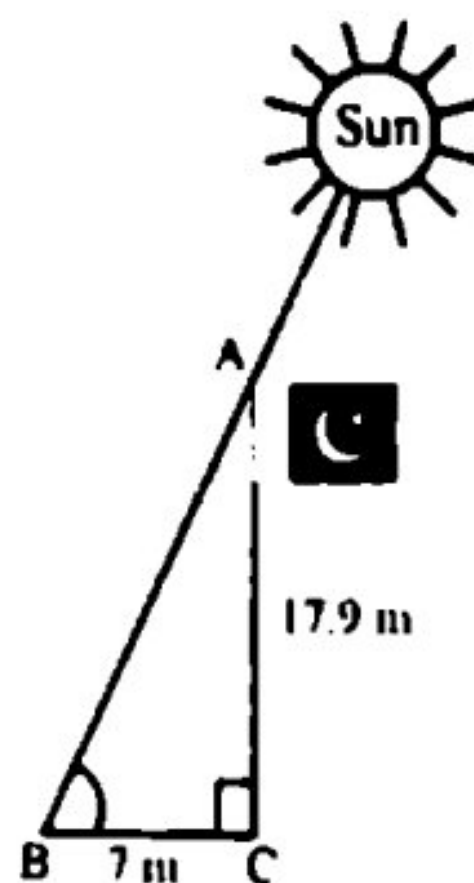


Find angle of elevation and angle of depression.

For finding distances height and angles by the use of trigonometric functions, consider the following examples:

Example 1

- A flagpole 17.9 meter high casts a 7 meter shadow. Find the angle of elevation of the sun.



Solution

From the figure, we observe that is the angle of elevation.
Using the fact that

$$\tan \alpha = \frac{AC}{BC} = \frac{17.9}{7} = 2.55714$$

Solving for α gives us

$$\alpha = \tan^{-1}(2.55714)$$

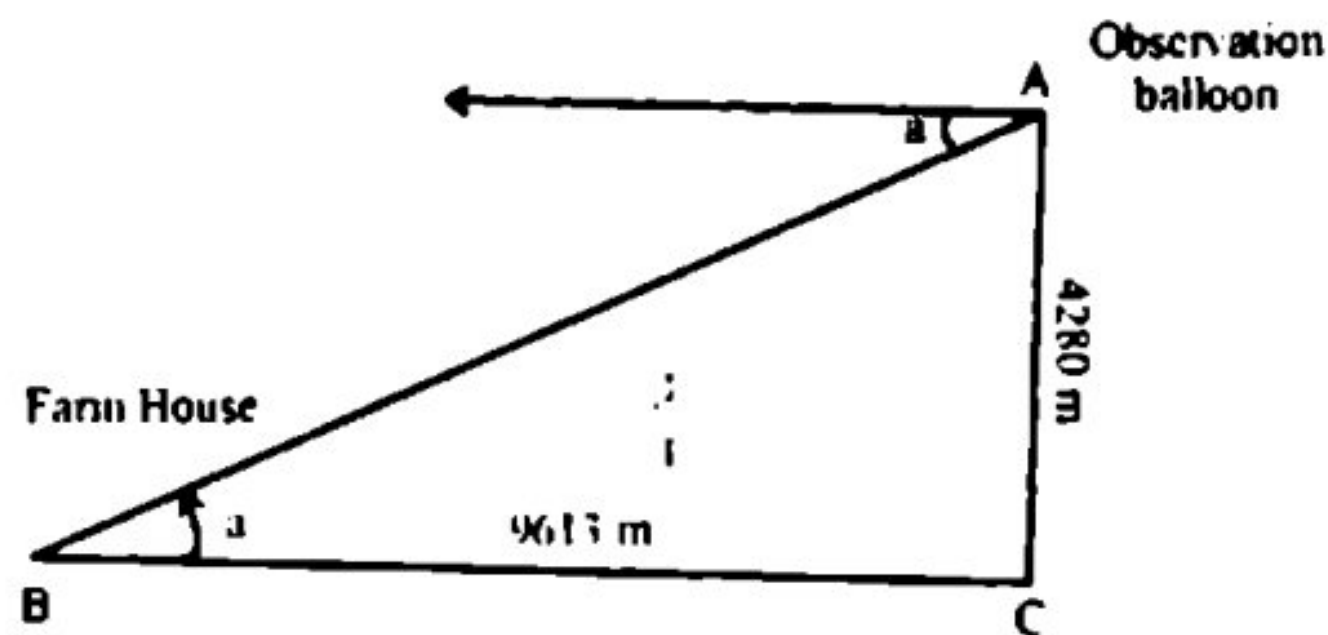
$$= (68.6666)^{\circ} = 68^{\circ}40'$$

$$\Rightarrow \alpha = 68^{\circ}40'$$

Example 2

An observation balloon is 4280 meter above the ground and 9613 meter away from a farmhouse. Find the angle of depression of the farmhouse as observed from the observation balloon.

Solution



For problems of this type the angle of elevation of A from B is considered equal to the angle of depression of B from A, as shown in the diagram.

$$\tan \alpha = \frac{AC}{BC} = \frac{4280}{9613} = 0.44523$$

$$\alpha = \tan^{-1}(0.44523) = 24^{\circ}$$

So, angle of depression is 24° .

SOLVED EXERCISE 7.5

- Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.

Solution

From the figure, we observe that α is the angle of elevation.

Using the fact that

$$\tan \alpha = \frac{BC}{AC}$$

$$\tan \alpha = \frac{6}{3.5}$$