

EXERCISE 1.1

Q.1 Given that:

(a) $f(x) = x^2 - x$

(b) $f(x) = \sqrt{x+4}$

Find: (i) $f(-2)$

(ii) $f(0)$

(iii) $f(x-1)$

(iv) $f(x^2+4)$

Solution:

(a) $f(x) = x^2 - x$

(i) $f(-2) = (-2)^2 - (-2)$
 $= 4 + 2 = 6$ Ans.

(ii) $f(0) = (0)^2 - 0$
 $= 0$ Ans.

(iii) $f(x-1) = (x-1)^2 - (x-1)$
 $= x^2 - 2x + 1 - x + 1$
 $= x^2 - 3x + 2$ Ans.

(iv) $f(x^2+4) = (x^2+4)^2 - (x^2+4)$
 $= x^4 + 8x^2 + 16 - x^2 - 4$
 $= x^4 + 7x^2 + 12$ Ans.

(b) $f(x) = \sqrt{x+4}$

(i) $f(-2) = \sqrt{-2+4} = \sqrt{2}$ Ans.

(ii) $f(0) = \sqrt{0+4} = \sqrt{4} = 2$ Ans.

(iii) $f(x-1) = \sqrt{x-1+4} =$

(iv) $f(x^2+4) = \sqrt{x^2+4+4} = \sqrt{x^2+8}$ Ans.

Q.2 Find $\frac{f(a+h)-f(a)}{h}$ and simplify where,

(i) $f(x) = 6x - 9$

(ii) $f(x) = \sin x$

(iii) $f(x) = x^3 + 2x^2 - 1$

(iv) $f(x) = \cos x$

Solution:

(i) $f(x) = 6x - 9$

$f(a+h) = 6(a+h) - 9$

$f(a+h) = 6a + 6h - 9$

$f(a) = 6a - 9$

$$\begin{aligned}
 \frac{f(a+h) - f(a)}{h} &= \frac{6a + 6h - 9 - (6a - 9)}{h} \\
 &= \frac{6a + 6h - 9 - 6a + 9}{h} \\
 &= \frac{6h}{h} \\
 &= 6 \text{ Ans.}
 \end{aligned}$$

(ii) $f(x) = \sin x$ (Lahore Board 2008)

$$f(a+h) = \sin(a+h)$$

$$f(a) = \sin a$$

$$\begin{aligned}
 \frac{f(a+h) - f(a)}{h} &= \frac{\sin(a+h) - \sin a}{h} \\
 &= \frac{2 \cos\left(\frac{a+h+a}{2}\right) \sin\left(\frac{a+h-a}{2}\right)}{h} \quad \because \sin p - \sin q = 2 \cos\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right) \\
 &= \frac{2}{h} \cos\left(\frac{2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \\
 &= \frac{2}{h} \cos\left(\frac{2a}{2} + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right) \\
 &= \frac{2}{h} \cos\left(a + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right) \text{ Ans.}
 \end{aligned}$$

(iii) $f(x) = x^3 + 2x^2 - 1$

$$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$$

$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2(a^2 + 2ah + h^2) - 1 \quad \because [(a+b)^3 - a^3 + b^3 + 3a^2b + 3ab^2]$$

$$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1$$

$$f(a) = a^3 + 2a^2 - 1$$

$$\begin{aligned}
 \frac{f(a+h) - f(a)}{h} &= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - (a^3 + 2a^2 - 1)}{h} \\
 &= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 4ah + 2h^2 - 1 - a^3 - 2a^2 + 1}{h} \\
 &= \frac{h^3 + 3a^2h + 3ah^2 + 4ah + 2h^2}{h} \\
 &= \frac{h(h^2 + 3a^2 + 3ah + 4a - 2h)}{h} \\
 &= h^2 + (3a+2)h + 3a^2 - 4a \text{ Ans.}
 \end{aligned}$$

$$(iv) \quad f(x) = \cos x$$

$$f(a+h) = \cos(a+h)$$

$$f(a) = \cos a$$

$$\frac{f(a+h) - f(a)}{h} = \frac{\cos(a+h) - \cos a}{h}$$

$$= \frac{-2 \sin\left(\frac{a+h+a}{2}\right) \sin\left(\frac{a+h-a}{2}\right)}{h}$$

$$\because \left[\cos p - \cos q = -2 \sin\left(\frac{p+q}{2}\right) \sin\left(\frac{p-q}{2}\right) \right]$$

$$= \frac{-2}{h} \sin\left(\frac{2a+h}{2}\right) \sin\left(\frac{h}{2}\right)$$

$$= \frac{-2}{h} \sin\left(\frac{2a}{2} + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)$$

$$= \frac{-2}{h} \sin\left(a + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right) \quad \text{Ans.}$$

Q.3 Express the following: (Lahore Board 2009-2010)

(a) The perimeter **P** of square as a function of its area **A**.

(b) The area **A** of a circle as a function of its circumference **C**.

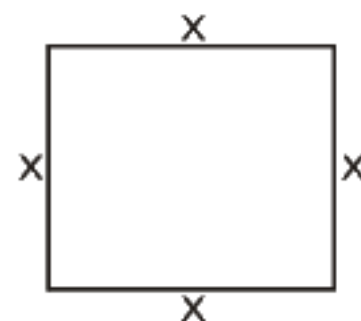
(c) The volume **V** of a cube as a function of the area **A** of its base.

Solution:

(a) Let,

$$\text{Length of square} = x$$

$$\text{Width of square} = x$$



$$\text{Perimeter of a square} = P = x + x + x + x$$

$$P = 4x \quad \dots\dots\dots (1)$$

$$\text{Area of a square} = A = x \times x$$

$$A = x^2$$

$$x = \sqrt{A}$$

Put $x = \sqrt{A}$ in equation (1)

$$\boxed{P = 4\sqrt{A}}$$

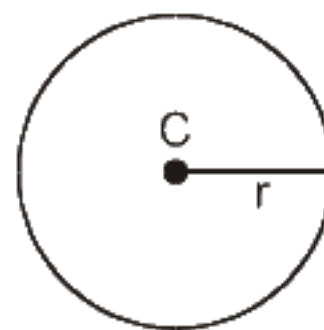
Shows perimeter **P** of a square as a function of its area **A**.

- (b) Let 'r' be the radius of the circle.

$$\text{Area of a circle} = A = \pi r^2 \quad \dots\dots\dots (1)$$

$$\text{Circumference of a circle} = C = 2\pi r$$

$$r = \frac{C}{2\pi}$$



$$\text{Put } r = \frac{C}{2\pi} \text{ in equation (1)}$$

$$A = \pi \left(\frac{C}{2\pi} \right)^2 = \frac{\pi C^2}{4\pi^2}$$

$$A = \frac{C^2}{4\pi}$$

Shows area A of a circle as a function of its circumference C.

- (c) Let x be the each side of cube. **(Gujranwala Board 2008)**

$$\text{Volume of cube} = V = x \times x \times x$$

$$V = x^3 \quad \dots\dots\dots (1)$$

$$\text{Area of base} = A = x \times x$$

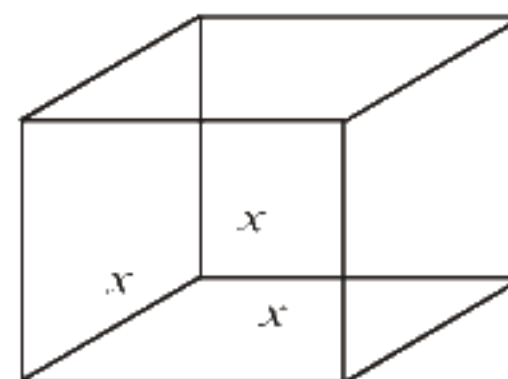
$$A = x^2$$

$$x = \sqrt{A}$$

$$\text{Put, } x = \sqrt{A} \text{ in equation (1)}$$

$$V = (\sqrt{A})^3$$

$$V = A^{3/2}$$



Shows volume V of a cube as a function of the area A of its base.

Q.4 Find the domain and the range of the function g defined below and sketch of graph of g.

(i) $g(x) = 2x - 5$

(ii) $g(x) = \sqrt{x^2 - 4}$

(iii) $g(x) = \sqrt{x+1}$ (Lhr.Board-2011)

(iv) $g(x) = |x - 3|$

(v) $g(x) = \begin{cases} 6x + 7, & x \leq -2 \\ x - 3, & x > -2 \end{cases}$

(vi) $g(x) = \begin{cases} x - 1, & x < 3 \\ 2x + 1, & 3 \leq x \end{cases}$

(vii) $g(x) = \frac{x^2 + 3x + 2}{x + 1}, x \neq -1$

(viii) $g(x) = \frac{x^2 - 16}{x - 4}, x \neq 4$

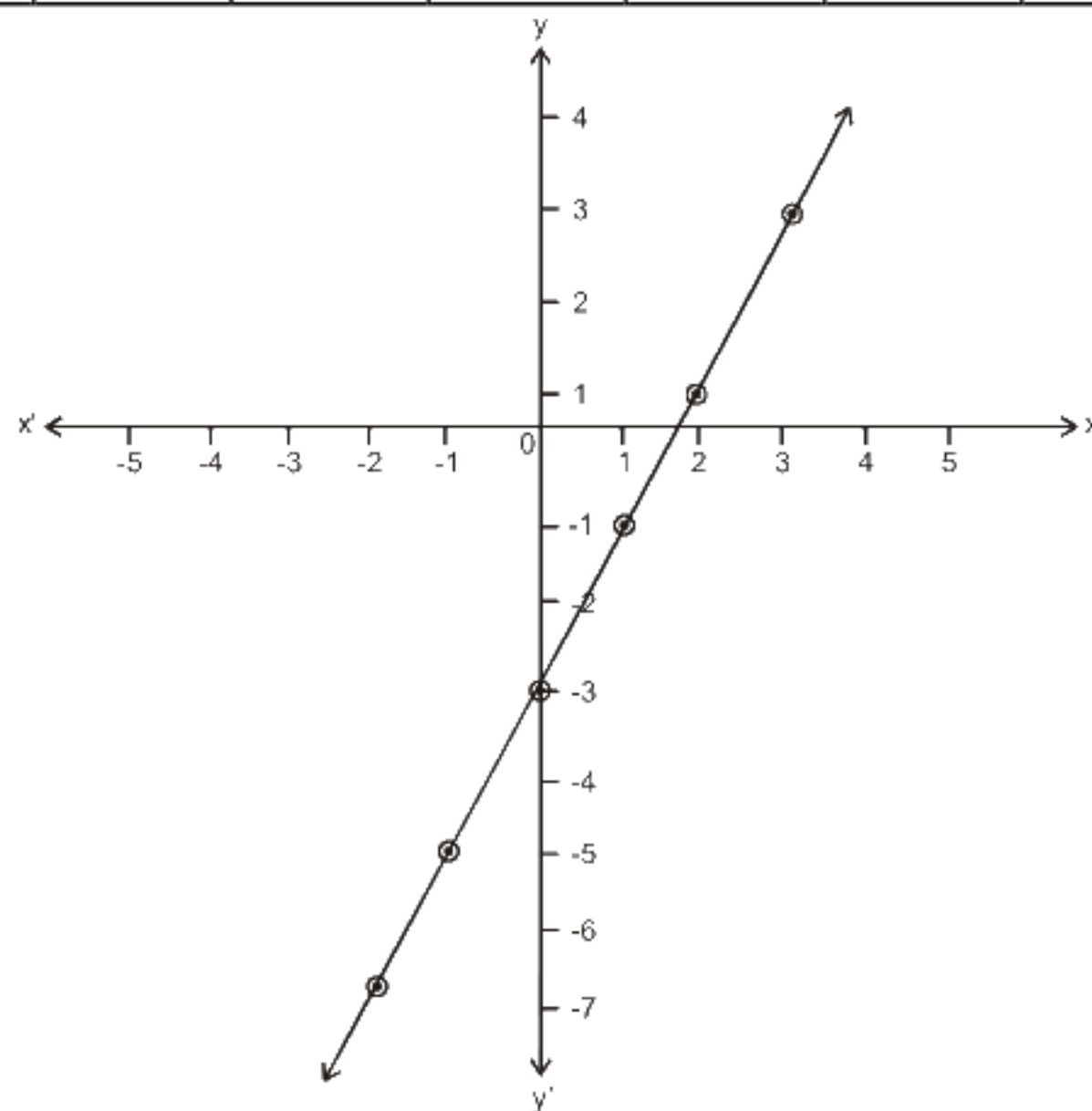
Solution:

(i) $g(x) = 2x - 5$

Domain of $g(x)$ = Set of all real numbers

Range of $g(x)$ = Set of all real numbers

x	-3	-2	-1	0	1	2	3
$g(x) = 2x - 5$	-7	-9	-7	-5	-3	-1	1

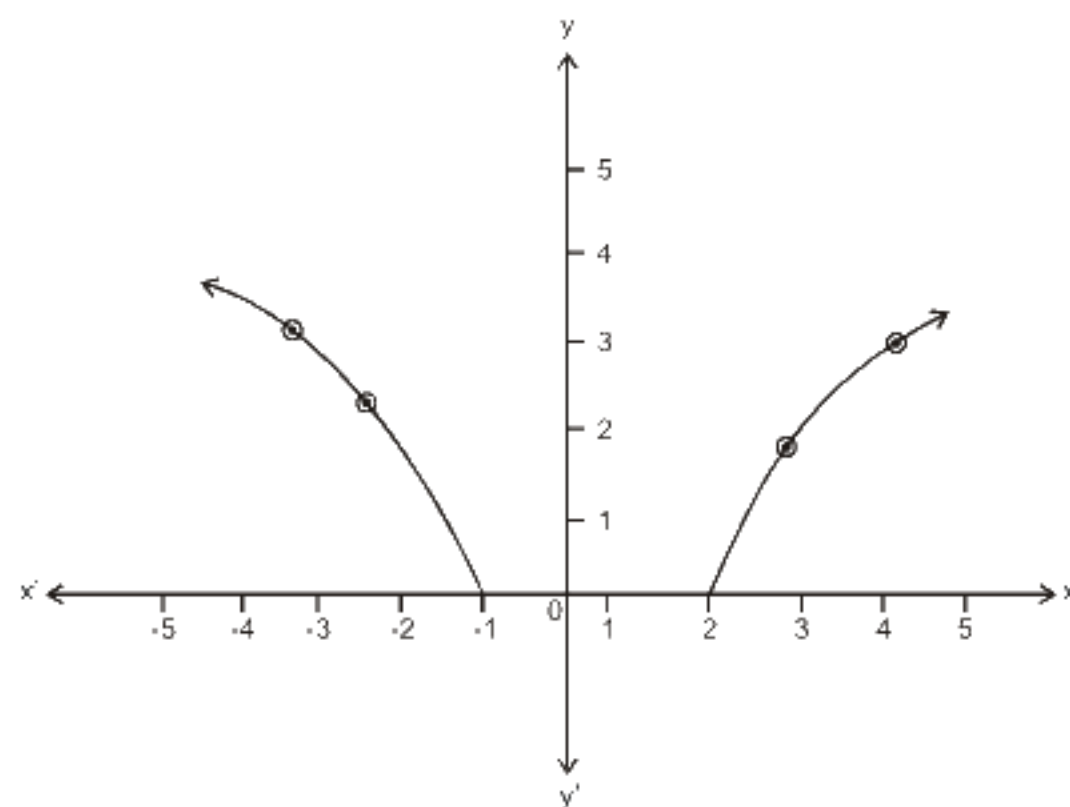


(ii) $g(x) = \sqrt{x^2 - 4}$ (Lahore Board 2008)

Domain of $g(x) = \mathbb{R} - (-2, 2)$

Range of $g(x) = [0, +\infty)$

x	-4	-3	-2	2	3	4
$g(x) = \sqrt{x^2 - 4}$	$2\sqrt{3}$	$\sqrt{5}$	0	0	$\sqrt{5}$	$2\sqrt{3}$

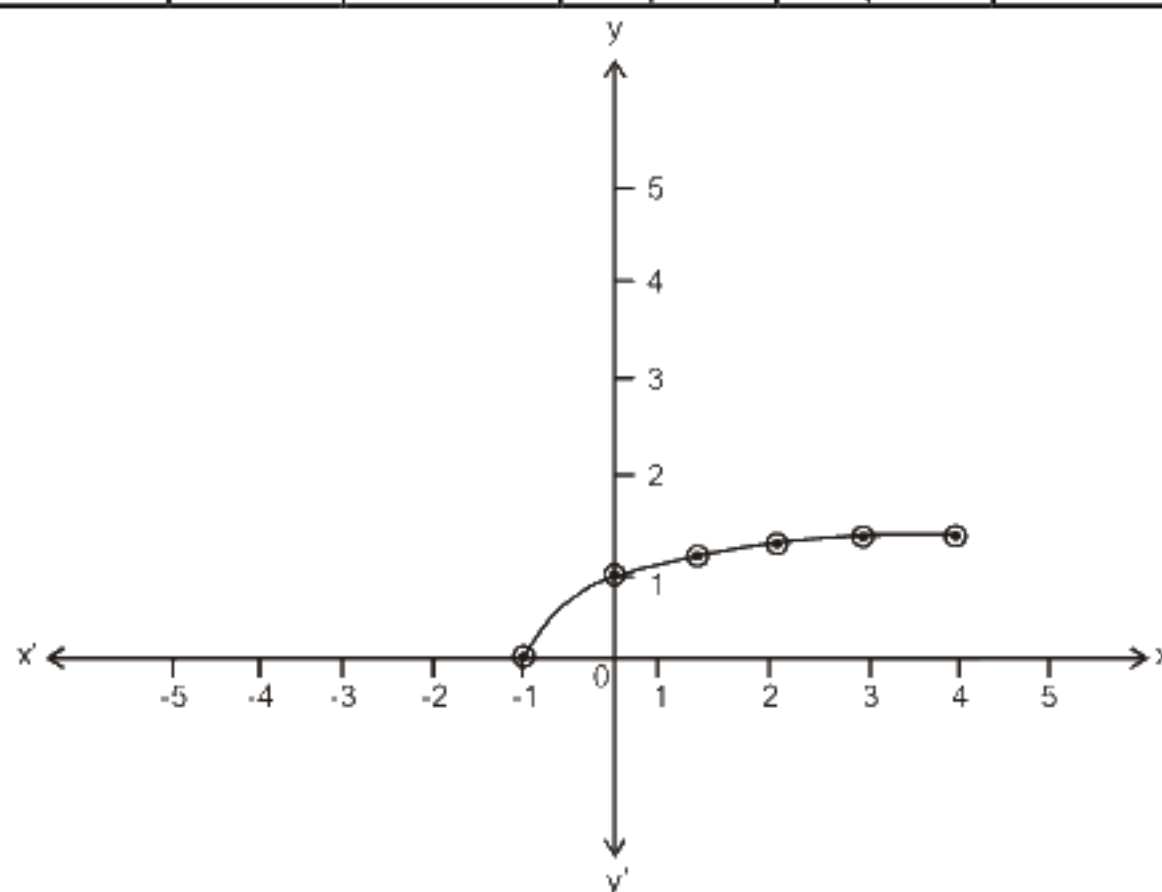


(iii) $g(x) = \sqrt{x+1}$

Domain of $g(x) = [-1, +\infty)$

Range of $g(x) = [0, +\infty)$

x	-1	0	1	2	3	4
$g(x) = \sqrt{x+1}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$

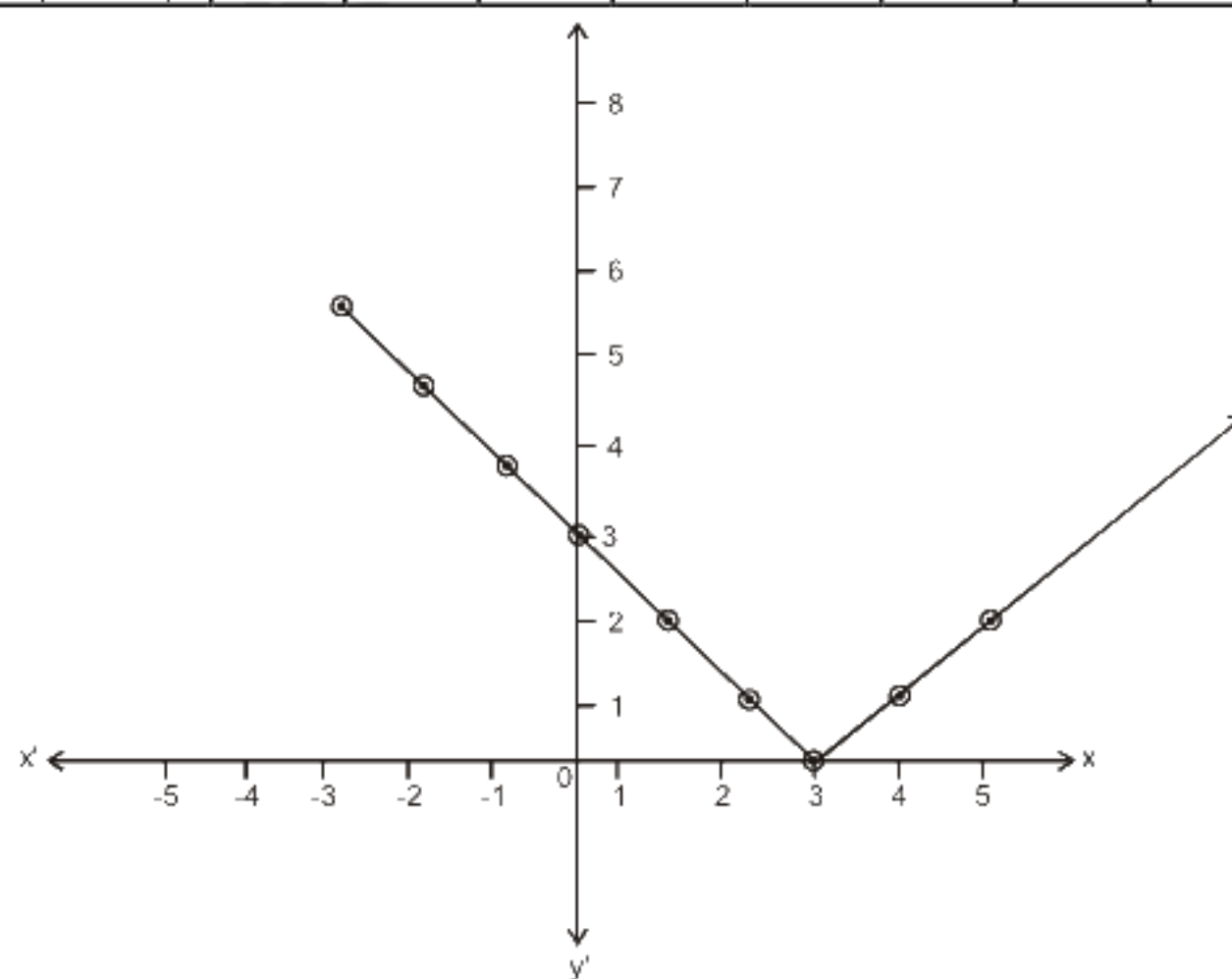


(iv) $g(x) = |x-3|$

Domain of $g(x) = (-\infty, +\infty)$

Range of $g(x) = [0, +\infty)$

x	-3	-2	-1	0	1	2	3	4	5
$g(x) = x-3 $	6	5	4	3	2	1	0	1	2



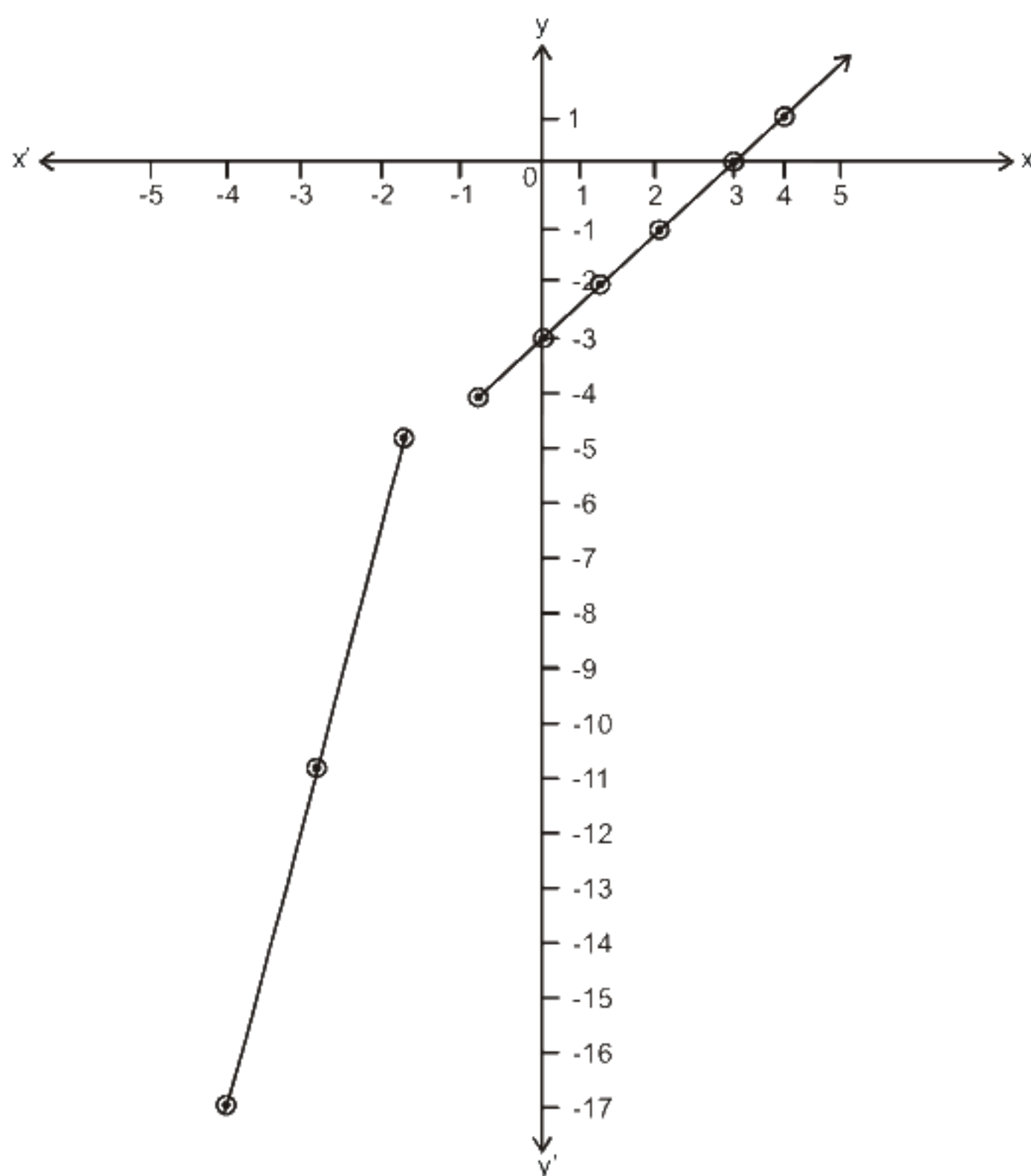
$$(v) \quad g(x) = \begin{cases} 6x + 7, & x \leq -2 \\ x - 3, & x > -2 \end{cases}$$

Domain of $g(x) = (-\infty, +\infty)$

Range of $g(x) = (-\infty, +\infty)$

$x \leq -2$	-2	-3	-4	-5
$g(x) = 6x + 7$	-5	-11	-17	-23

$x > -2$	-1	0	1	2	3	3
$g(x) = x - 3$	-4	-3	-2	-1	0	1



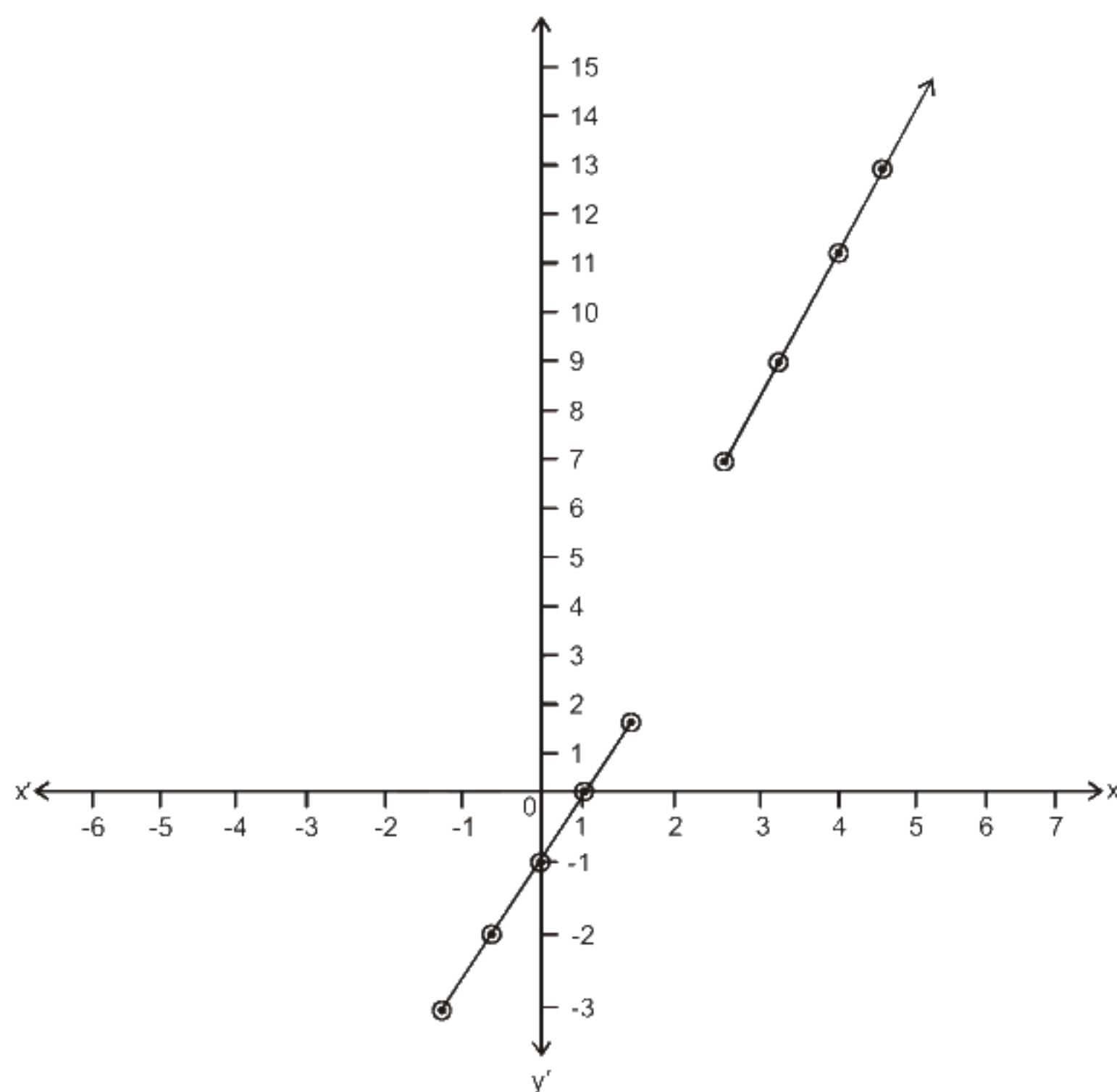
$$(vi) \quad g(x) = \begin{cases} x - 1, & x < 3 \\ 2x + 1, & 3 \leq x \end{cases}$$

Domain of $g(x) = (-\infty, +\infty)$

Range of $g(x) = (-\infty, 2) \cup [7, +\infty)$

$x < 3$	-2	-1	0	1	2
$g(x) = x - 1$	-3	-2	-1	0	1

$x \geq 3$	3	4	5	6
$g(x) = 2x + 1$	7	9	11	13

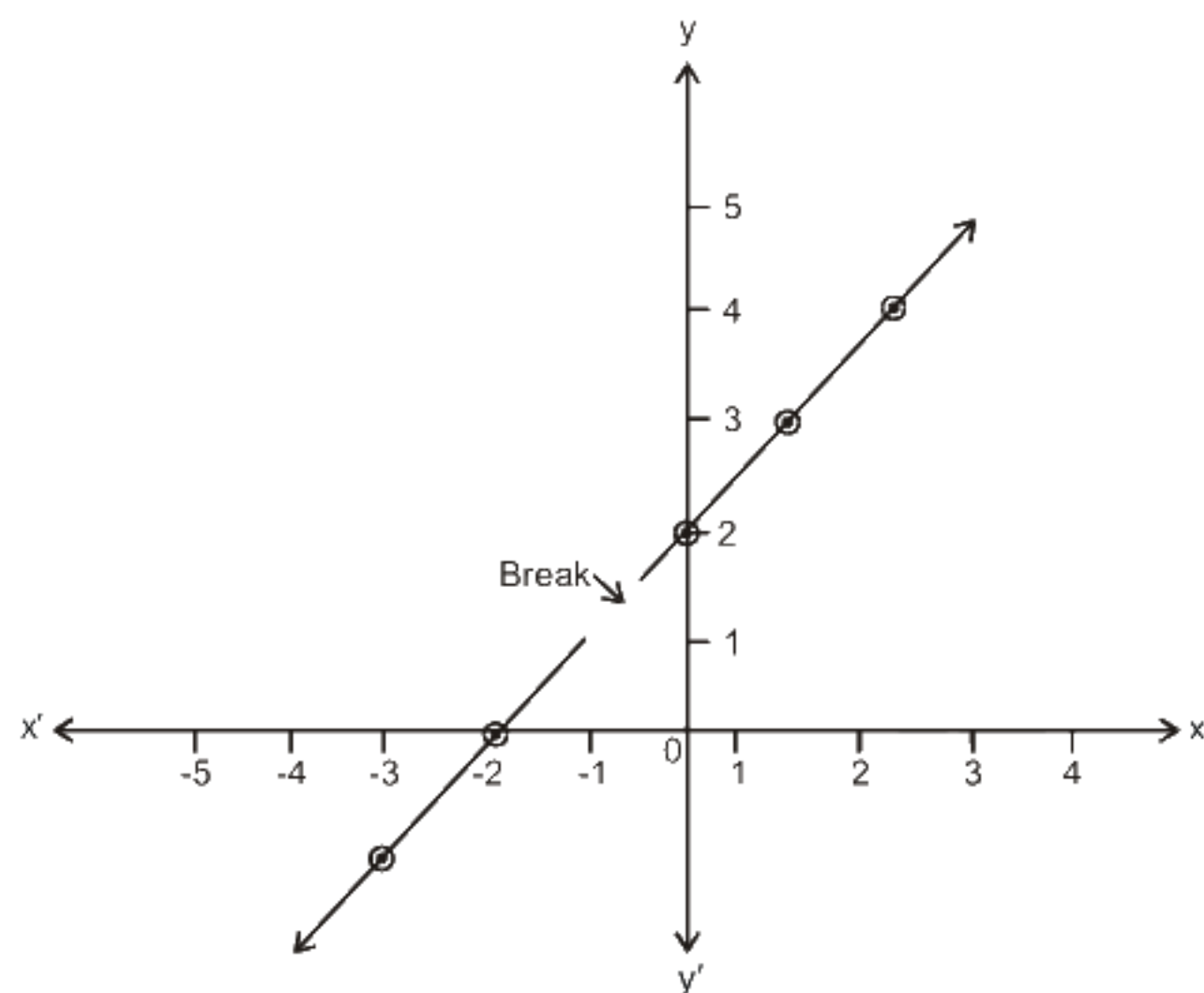


$$\begin{aligned}
 \text{(vii)} \quad g(x) &= \frac{x^2 + 3x + 2}{x + 1}, \quad x \neq -1 \\
 g(x) &= \frac{x^2 + 2x + x + 2}{x + 1} \\
 &= \frac{x(x + 2) + 1(x + 2)}{x + 1} \\
 &= \frac{(x + 2)(x + 1)}{x + 1} = x + 2
 \end{aligned}$$

$$\text{Domain of } g(x) = \mathbb{R} - \{-1\}$$

$$\text{Range of } g(x) = \mathbb{R} - \{1\}$$

x	-3	-2	0	1	2
$g(x) = x + 2$	-1	0	2	3	4



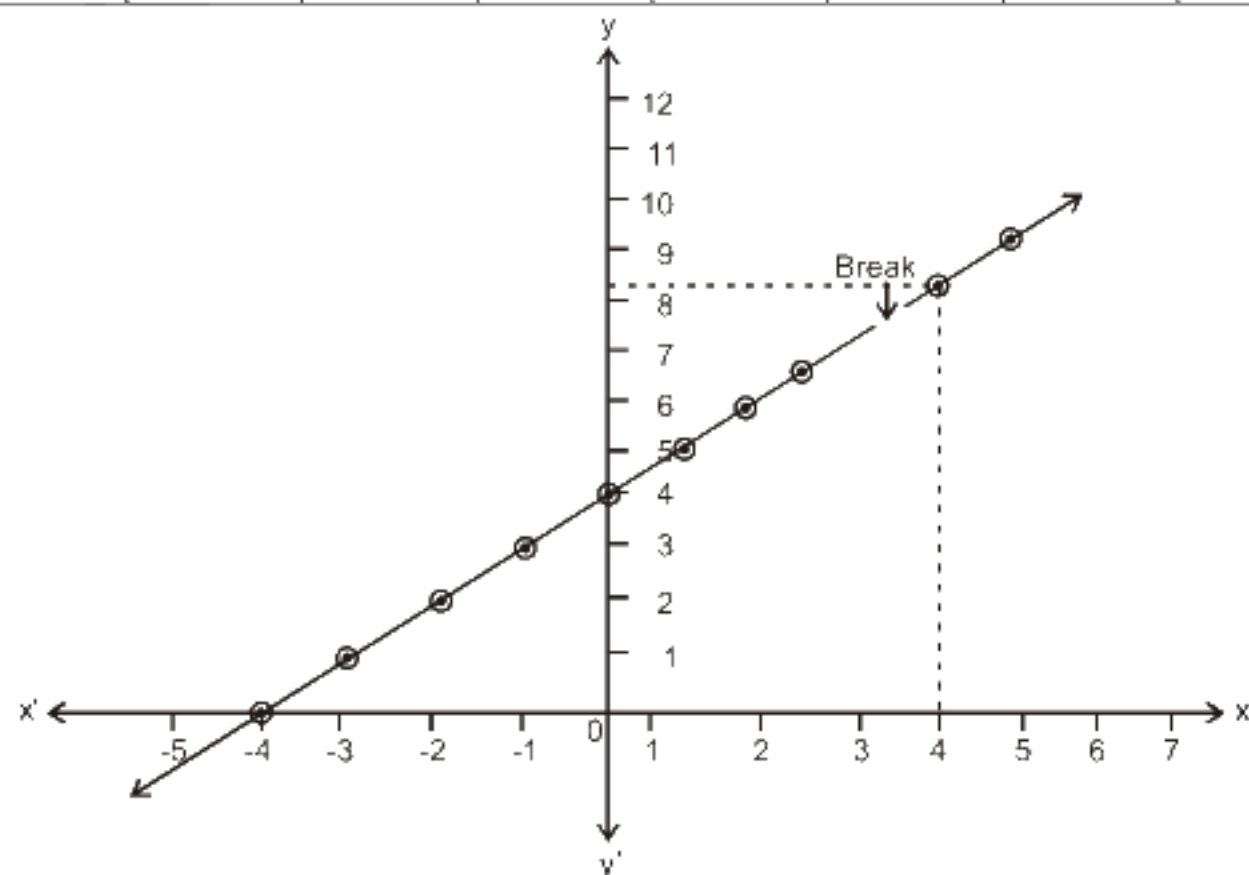
(viii)
$$g(x) = \frac{x^2 - 16}{x - 4}, \quad x \neq 4$$

$$= \frac{(x + 4)(x - 4)}{x - 4} = x + 4$$

Domain of $g(x) = \mathbb{R} - \{4\}$

Range of $g(x) = \mathbb{R} - \{8\}$

x	-4	-3	-2	-1	0	1	2	3	5	6
$g(x) = x + 4$	0	1	2	3	4	5	6	7	9	10



Q.5 Given $f(x) = x^3 - ax^2 + bx + 1$.

If $f(2) = -3$ and $f(-1) = 0$. Find the values of a and b .

Solution:

$$f(x) = x^3 - ax^2 + bx + 1 \quad \dots\dots\dots (1)$$

Put $x = 2$ in equation (1)

$$\begin{aligned} f(2) &= (2)^3 - a(2)^2 + b(2) + 1 \\ &= 8 - 4a + 2b + 1 \end{aligned}$$

$$f(2) = 9 - 4a + 2b$$

Put $x = -1$ in equation (1)

$$\begin{aligned} f(-1) &= (-1)^3 - a(-1)^2 + b(-1) + 1 \\ &= -1 - a - b + 1 \end{aligned}$$

$$f(-1) = -a - b$$

$$\text{Since } f(2) = -3$$

$$9 - 4a + 2b = -3$$

$$-4a + 2b = -3 - 9$$

$$-2(2a - b) = -12$$

$$2a - b = \frac{-12}{-2}$$

$$2a - b = 6 \quad \dots\dots\dots (2)$$

And

$$f(-1) = 0$$

$$-a - b = 0$$

$$-a = b$$

$$a = -b \quad \dots\dots\dots (3)$$

Put $a = -b$ in equation (2)

$$2(-b) - b = 6$$

$$-2b - b = 6$$

$$-3b = 6$$

$$b = \frac{6}{-3} = -2$$

Put $b = -2$ in equation (2)

$$a = -(-2)$$

$$a = 2$$

$\therefore$ $a = 2, b = -2$ Ans.

Q.6 A stone falls from a height of 60m on the ground, the height h after x second is approximately given by $h(x) = 40 - 10x^2$.

(i) What is the height of the stone when.

(a) $x = 1$ sec (b) $x = 1.5$ sec (c) $x = 1.7$ sec

(ii) When does the stone strike the ground?

Solution:

(i) $h(x) = 40 - 10x^2$ (1)

(a) Put $x = 1$ sec in equation (1)

$$\begin{aligned} h(1) &= 40 - 10(1)^2 \\ &= 40 - 10 \\ &= 30\text{m} \quad \text{Ans.} \end{aligned}$$

(b) Put $x = 1.5$ sec in equation (1)

$$\begin{aligned} h(1.5) &= 40 - 10(1.5)^2 \\ &= 40 - 10(2.25) \\ &= 40 - 22.5 \\ &= 17.5\text{m} \quad \text{Ans.} \end{aligned}$$

(c) Put $x = 1.7$ sec in equation (1)

$$\begin{aligned} h(1.7) &= 40 - 10(1.7)^2 \\ &= 40 - 10(2.89) \\ &= 40 - 28.9 \\ &= 11.1 \text{ m} \quad \text{Ans.} \end{aligned}$$

(ii) When then the stone strike the ground.

then $h(x) = 0$

$$0 = 40 - 10x^2$$

$$10x^2 = 40$$

$$x^2 = \frac{40}{10}$$

$$x^2 = 4$$

$$x = 2 \text{ sec} \quad \text{Ans.}$$

Q.7: Show that the Parametric equations.

(i) $x = at^2$, $y = 2at$ represent the equation of Parabola $y^2 = 4ax$

(ii) $x = a\cos\theta$, $y = b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(iii) $x = a\sec\theta$, $y = b\tan\theta$ represent the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:

(i) $x = at^2$ (i) , $y = 2at$ (ii)

From equation (ii)

$$t = \frac{y}{2a}$$

Putting it in (i)

$$x = a \left(\frac{y}{2a} \right)^2 = a \left(\frac{y^2}{4a^2} \right)$$

$$y^2 = 4ax \quad \text{Hence proved.}$$

$$(ii) \quad x = a \cos \theta$$

$$\frac{x}{a} = \cos \theta$$

Squaring on both sides

$$\frac{x^2}{a^2} = \cos^2 \theta \quad \dots (i)$$

$$y = b \sin \theta$$

$$\frac{y}{b} = \sin \theta$$

Squaring on both sides

$$\frac{y^2}{b^2} = \sin^2 \theta \quad \dots (ii)$$

Adding equation (i) & equation (ii)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \theta + \sin^2 \theta$$

$$= 1 \quad \text{Hence proved.}$$

$$(iii) \quad x = a \sec \theta$$

$$\frac{x}{a} = \sec \theta$$

Squaring on both sides

$$\frac{x^2}{a^2} = \sec^2 \theta \quad \dots (i)$$

$$y = b \tan \theta$$

$$\frac{y}{b} = \tan \theta$$

Squaring on both sides

$$\frac{y^2}{b^2} = \tan^2 \theta \quad \dots (ii)$$

Subtracting equation (ii) from equation (i)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \theta - \tan^2 \theta$$

$$= 1 + \tan^2 \theta - \tan^2 \theta$$

$$= 1 \quad \text{Hence proved.}$$

Q.8 Prove the identities:

$$(i) \quad \sinh 2x = 2 \sinh x \cosh x$$

(Lahore Board 2006)

$$(ii) \quad \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$(iii) \quad \operatorname{cosech}^2 x = \coth^2 x - 1$$

Solution:

$$(i) \quad \sinh 2x = 2 \sinh x \cosh x$$

$$\text{R.H.S} = 2 \sinh x \cosh x$$

$$\begin{aligned}
 &= 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \frac{e^{2x} - e^{-2x}}{2} \\
 &= \sinh 2x \\
 &= \text{L.H.S.} \quad \text{Hence proved.}
 \end{aligned}$$

(ii) **$\sec h^2 x = 1 - \tanh^2 x$**

$$\begin{aligned}
 \text{R.H.S} &= 1 - \tanh^2 x \\
 &= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 \\
 &= 1 - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{e^{2x} + e^{-2x} + 2e^x \cdot e^{-x} - (e^{2x} + e^{-2x} - 2e^x \cdot e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2} \\
 &= \frac{4}{(e^x + e^{-x})^2} \\
 &= \left(\frac{2}{e^x + e^{-x}} \right)^2 \\
 &= (\sec hx)^2 \\
 &= \text{sech}^2 x \\
 &= \text{L.H.S} \quad \text{Hence proved}
 \end{aligned}$$

(iii) **$\operatorname{cosec} h^2 x = \cot h^2 x - 1$**

$$\begin{aligned}
 \text{R.H.S} &= \cot h^2 x - 1 \\
 &= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 - 1 \\
 &= \frac{(e^x + e^{-x})^2}{(e^x - e^{-x})^2} - 1 \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2} \\
 &= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x \cdot e^{-x})}{(e^x - e^{-x})^2}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x - e^{-x})^2} \\
&= \frac{4}{(e^x + e^{-x})^2} \\
&= \left(\frac{2}{e^x - e^{-x}} \right)^2 \\
&= (\operatorname{cosech} x)^2 = \operatorname{cosech}^2 x \\
&= \text{L.H.S} \quad \text{Hence proved}
\end{aligned}$$

Q.9 Determine whether the given function f is even or odd:

(i) $f(x) = x^3 + x$

(ii) $f(x) = (x + 2)^2$

(iii) $f(x) = x\sqrt{x^2 + 5}$

(iv) $f(x) = \frac{x-1}{x+1}, x \neq -1$

(v) $f(x) = x^{2/3} + 6$

(vi) $f(x) = \frac{x^3 - x}{x^2 + 1}$

Solution:

(i) $f(x) = x^3 + x$

$$\begin{aligned}
f(-x) &= (-x)^3 + (-x) \\
&= -x^3 - x \\
&= -(x^3 + x) \\
&= -f(x)
\end{aligned}$$

$\therefore f(x)$ is an odd function.

(ii) $f(x) = (x + 2)^2$

$$\begin{aligned}
f(-x) &= (-x + 2)^2 \\
&\neq \pm f(x)
\end{aligned}$$

$\therefore f(x)$ is neither even nor odd function.

(iii) $f(x) = x\sqrt{x^2 + 5}$

$$\begin{aligned}
f(-x) &= -x\sqrt{(-x)^2 + 5} \\
&= -x\sqrt{x^2 + 5} \\
&= -f(x)
\end{aligned}$$

$\therefore f(x)$ is an odd function.

(iv) $f(x) = \frac{x-1}{x+1}, x \neq -1$

$$f(-x) = \frac{-x-1}{-x+1} = \frac{-(x+1)}{-(x-1)}$$

$$= \frac{x+1}{x-1} \neq \pm f(x)$$

$\therefore f(x)$ is neither even nor odd function.

(v) $f(x) = x^{2/3} + 6$

$$\begin{aligned} f(-x) &= (-x)^{2/3} + 6 \\ &= [(-x)^2]^{1/3} + 6 \\ &= (x^2)^{1/3} + 6 \\ &= x^{2/3} + 6 \\ &= f(x) \end{aligned}$$

$\therefore f(x)$ is an even function.

(vi) $f(x) = \frac{x^3 - x}{x^2 + 1}$

$$\begin{aligned} f(-x) &= \frac{(-x)^3 - (-x)}{(-x)^2 - 1} \\ &= \frac{-x^3 + x}{x^2 + 1} \\ &= \frac{-(x^3 - x)}{x^2 + 1} \\ &= -f(x) \end{aligned}$$

$\therefore f(x)$ is an odd function.

Composition of Functions:

Let f be a function from set X to set Y and g be a function from set Y to set Z . The composition of f and g is a function, denoted by $g \circ f$, from X to Z and is defined by.

$$(g \circ f)(x) = g(f(x)) = gf(x) \text{ for all } x \in X$$

Inverse of a Function:

Let f be one-one function from X onto Y . The inverse function of f , denoted by f^{-1} , is a function from Y onto X and is defined by.

$$x = f^{-1}(y), \forall y \in Y \text{ if and only if } y = f(x), \forall x \in X$$

EXERCISE 1.2

Q.1 The real valued functions f and g are defined below. Find

(a) $f \circ g(x)$ (b) $g \circ f(x)$ (c) $f \circ f(x)$ (d) $g \circ g(x)$

(i) $f(x) = 2x + 1$; $g(x) = \frac{3}{x-1}$, $x \neq 1$