Deduction

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

We know that

$$\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$$

Put $a = e$

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = \log_e e = 1$$

Important results to remember

(i) $\lim_{x \to +\infty} (e^x) = \infty$ (ii) $\lim_{x \to -\infty} (e^x) = \lim_{x \to -\infty} \left(\frac{1}{e^{-x}}\right) = 0$

(iii) $\lim_{x \to \pm\infty} \left(\frac{a}{x}\right) = 0$, where a is any real number.

EXERCISE 1.3

Q.1 Evaluate each limit by using theorems of limits.

(i) $\lim_{x\to 3} (2x + 4)$ **(ii)** $\lim_{x\to 1} (3x^2 - 2x + 4)$

(iii) $\lim_{x\to 3} \sqrt{x^2 + x + 4}$ **(iv)** $\lim_{x\to 2} x\sqrt{x^2 - 4}$

(v) $\lim_{x\to 2} \left(\sqrt{x^3 + 1} - \sqrt{x^2 + 5}\right)$ **(iv)** $\lim_{x\to 2} \frac{2x^3 + 5x}{3x - 2}$

Solution:

(i) $\lim_{x\to 3} (2x + 4) = \lim_{x\to 3} (2x) + \lim_{x\to 3} (4)$

$= 2 \lim_{x\to 3} x + 4$

$= 2(3) + 4 = 6 + 4 = 10$ Ans.

(ii) $\lim_{x\to 1} (3x^2 - 2x + 4) = \lim_{x\to 1} (3x^2) - \lim_{x\to 1} (2x) + \lim_{x\to 1} (4)$

$= 3 \lim_{x\to 1} x^2 - 2 \lim_{x\to 1} x + 4$

$= 3(1)^2 - 2(1) + 4$

$= 3 - 2 + 4$

$= 5$ Ans.

(iii) $\lim_{x\to 3} \sqrt{x^2 + x + 4} = [\lim_{x\to 3} (x^2 + x + 4)]^{1/2}$

$= [\lim_{x\to 3} x^2 + \lim_{x\to 3} x + \lim_{x\to 3} 4]^{1/2}$

$= (3^2 + 3 + 4)^{1/2}$

$= (9 + 7)^{1/2} = (16)^{1/2} = (4^2)^{1/2} = 4$ Ans.

(iv) $\lim_{x\to 2} x\sqrt{x^2-4} = [\lim_{x\to 2}(x)]\,[\lim_{x\to 2}(x^2-4)^{1/2}]$

$= 2\,[\lim_{x\to 2}(x^2-4)]^{1/2}$

$= 2\,[\lim_{x\to 2} x^2 - \lim_{x\to 2} 4]^{1/2}$

$= 2\,(4-4)^{1/2}$

$= 2(0)^{1/2}$

$= 0$ Ans.

(v) $\lim_{x\to 2}\left(\sqrt{x^3+1} - \sqrt{x^2+5}\right) = \lim_{x\to 2}(x^3+1)^{1/2} - \lim_{x\to 2}(x^2+5)^{1/2}$

$= [\lim_{x\to 2}(x^3+1)]^{1/2} - [\lim_{x\to 2}(x^2+5)]^{1/2}$

$= [\lim_{x\to 2} x^3 + \lim_{x\to 2} 1]^{1/2} - [\lim_{x\to 2} x^2 + \lim_{x\to 2} 5]^{1/2}$

$= (8+1)^{1/2} - (4+5)^{1/2}$

$= (9)^{1/2} - (9)^{1/2} = (3^2)^{1/2} - (3^2)^{1/2} = 3 - 3$

$= 0$ Ans.

(vi) $\lim_{x\to -2}\dfrac{2x^3+5x}{3x-2} = \dfrac{\lim_{x\to -2}(2x^3+5x)}{\lim_{x\to -2}(3x-2)}$

$= \dfrac{2\lim_{x\to -2} x^3 + 5\lim_{x\to -2} x}{3\lim_{x\to -2} x - \lim_{x\to -2} 2}$

$= \dfrac{2(-2)^3 + 5(-2)}{3(-2) - 2}$

$= \dfrac{2(-8) - 10}{-6-2} = \dfrac{-16-10}{-8} = \dfrac{-26}{-8} = \dfrac{13}{4}$ Ans

Q.2 Evaluate each limit by using algebraic techniques.

(i) $\lim_{x\to -1}\dfrac{x^3-x}{x+1}$ **(ii)** $\lim_{x\to 1}\left(\dfrac{3x^3+4x}{x^2+x}\right)$

(iii) $\lim_{x\to 2}\dfrac{x^3-8}{x^2+x-6}$ **(iv)** $\lim_{x\to 1}\dfrac{x^3-3x^2+3x-1}{x^3-x}$ **(Lhr. Board 2009)**

(v) $\lim_{x\to -1}\left(\dfrac{x^3+x^2}{x^2-1}\right)$ **(vi)** $\lim_{x\to 4}\dfrac{2x^2-32}{x^3-4x^2}$

(vii) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ **(Lhr. Board 2006)**

(viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ **(Lhr. Board 2004)** **(ix)** $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x^m-a^m}$

Solution:

(i) $\lim\limits_{x\to -1} \dfrac{x^3-x}{x+1}$ $\left(\dfrac{0}{0}\right)$ **form**

$$\lim_{x\to -1} \frac{x^3-x}{x+1} = \lim_{x\to -1} \frac{x(x^2-1)}{x+1}$$

$$= \lim_{x\to -1} \frac{x(x+1)(x-1)}{x+1}$$

$$= \lim_{x\to -1} x(x-1)$$

$$= -1(-1-1)$$

$$= -1(-2) = 2 \quad \text{Ans.}$$

(ii) $\lim\limits_{x\to 1} \left(\dfrac{3x^3+4x}{x^2+x}\right) = \dfrac{3(1)^3+4(1)}{(1)^2+1}$

$$= \frac{3+4}{2} = \frac{7}{2} \quad \text{Ans.}$$

(iii) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2+x-6}$ $\left(\dfrac{0}{0}\right)$ **form** ***(Gujranwala 2007, Lahore Board 2008)***

$$\lim_{x\to 2} \frac{x^3-8}{x^2+x-6} = \lim_{x\to 2} \frac{(x)^3-(2)^3}{x^2+3x-2x-6}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x(x+3)-2(x+3)} \quad [\because\ a^3-b^3=(a-b)(a^2+ab+b^2)]$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x+3)(x-2)}$$

$$= \frac{(2)^2+2(2)+4}{2+3} = \frac{4+4+4}{5} = \frac{12}{5} \quad \text{Ans.}$$

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^3-x}$ $\left(\dfrac{0}{0}\right)$ **form** ***(Lahore Board 2009)***

$$\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^3-x} = \lim_{x\to 1} \frac{(x-1)^3}{x(x^2-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)^3}{x(x+1)(x-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)^2}{x(x+1)}$$

$$= \frac{(1-1)^2}{1(1+1)} = \frac{0}{2} = 0 \qquad \text{Ans.}$$

(v) $\lim\limits_{x\to -1} \left(\dfrac{x^3+x^2}{x^2-1}\right)$ $\left(\dfrac{0}{0}\right)$ **form**

$$\lim_{x\to -1} \left(\frac{x^3+x^2}{x^2-1}\right) = \lim_{x\to -1} \frac{x^2(x+1)}{(x+1)(x-1)}$$

$$= \lim_{x\to -1} \frac{x^2}{x-1}$$

$$= \frac{(-1)^2}{-1-1} = \frac{-1}{2} \qquad \text{Ans.}$$

(vi) $\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2}$ $\left(\dfrac{0}{0}\right)$ **form**

$$\lim_{x\to 4} \frac{2x^2-32}{x^3-4x^2} = \lim_{x\to 4} \frac{2(x^2-16)}{x^2(x-4)}$$

$$= \lim_{x\to 4} \frac{2(x+4)(x-4)}{x^2(x-4)}$$

$$= \lim_{x\to 4} \frac{2(x+4)}{x^2}$$

$$= \frac{2(4+4)}{(4)^2} = \frac{2(8)}{16}$$

$$= \frac{16}{16} = 1 \qquad \text{Ans.}$$

(vii) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ $\left(\dfrac{0}{0}\right)$ **form** **(Guj. Board 2006)**

$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} \times \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x\to 2} \frac{(\sqrt{x})^2-(\sqrt{2})^2}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} \qquad \text{Ans.}$$

(viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ $\left(\dfrac{0}{0}\right)$ **form** ***(Lahore Board 2006)***

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \quad \text{Ans.}$$

(ix) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x^m-a^m}$ $\left(\dfrac{0}{0}\right)$ **form**

We know that:

$$\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}, \quad \text{where n is an integer and } a > 0$$

Now,

$$\lim_{x\to a} \frac{x^n-a^n}{x^m-a^m} = \lim_{x\to 0} \frac{\dfrac{x^n-a^n}{x-a}}{\dfrac{x^m-a^m}{x-a}}$$

$$= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m}a^{n-1-m-1} = \frac{n}{m}a^{n-m} \quad \text{Ans.}$$

Q.3 Evaluate the following limits:

(i) $\lim\limits_{x\to 0} \dfrac{\sin 7x}{x}$ **(ii)** $\lim\limits_{x\to 0} \dfrac{\sin x^{o}}{x}$ **(L.B 2003)**

(iii) $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta}$ **(L.B 2009 (s))** **(iv)** $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi - x}$

(v) $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx}$ **(vi)** $\lim\limits_{x\to 0} \dfrac{x}{\tan x}$

(vii) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x^2}$ **(viii)** $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin^2 x}$ **(L.B 2009)**

(ix) $\lim\limits_{\theta \to 0} \dfrac{\sin^2\theta}{\theta}$ **(L.B 2007)** **(x)** $\lim\limits_{x \to 0} \dfrac{\sec x - \cos x}{x}$

(xi) $\lim\limits_{\theta \to 0} \dfrac{1 - \cos p\theta}{1 - \cos q\theta}$ **(L.B 2004,06)** **(G.B 2005, 2006)**

(xii) $\lim\limits_{\theta \to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$ **(L.B 2003, 2004) (G.B 2005)**

Solution:

(i) $\lim\limits_{x \to 0} \dfrac{\sin 7x}{x}$ $\left(\dfrac{0}{0}\right)$ **form**

$$\lim_{x \to 0} \frac{\sin 7x}{x} = \lim_{x \to 0} \frac{\sin 7x}{7x} \times 7$$

$$= 1 \times 7 = 7 \quad \text{Ans.}$$

(ii) $\lim\limits_{x \to 0} \dfrac{\sin x^o}{x}$ $\left(\dfrac{0}{0}\right)$ **from**

$$\lim_{x \to 0} \frac{\sin x^o}{x} = \lim_{x \to 0} \frac{\sin \frac{x\pi}{180}}{\frac{x\pi}{180}} \times \frac{\pi}{180}$$

$$= 1 \times \frac{\pi}{180} = \frac{\pi}{180} \quad \text{Ans.}$$

$\because 1^\circ = \dfrac{\pi}{180}$ radian

$x^\circ = \dfrac{x\pi}{180}$ radian

(iii) $\lim\limits_{\theta \to 0} \dfrac{1 - \cos\theta}{\sin\theta}$ $\left(\dfrac{0}{0}\right)$ **form**

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta} = \lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\sin\theta\,(1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin^2\theta}{\sin\theta\,(1 + \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{0}{1 + 1} = \frac{0}{2} = 0 \quad \text{Ans.}$$

(iv) $\displaystyle \lim_{x\to\pi} \frac{\sin x}{\pi - x}$ $\left(\frac{0}{0}\right)$ **form**

Put $\pi - x = t \Rightarrow x = \pi - t$

As $x \to \pi$, $t \to 0$

$$\lim_{x\to\pi} \frac{\sin x}{\pi - x} = \lim_{t\to 0} \frac{\sin(\pi - t)}{t}$$

$$= \lim_{t\to 0} \frac{\sin t}{t}$$

$$= 1$$ Ans.

(v) $\displaystyle \lim_{x\to 0} \frac{\sin ax}{\sin bx}$ $\left(\frac{0}{0}\right)$ **form** ***(G.B 2007)***

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$$

$$= \lim_{x\to 0} \frac{\frac{\sin ax}{ax} \times a}{\frac{\sin bx}{bx} \times b}$$

$$= \frac{1 \times a}{1 \times b} = \frac{a}{b}$$ Ans.

(vi) $\displaystyle \lim_{x\to 0} \frac{x}{\tan x}$ $\left(\frac{0}{0}\right)$ **form** ***(L.B 2008)***

$$\lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{x}{\frac{\sin x}{\cos x}}$$

$$= \lim_{x\to 0} \frac{\cos x}{\frac{\sin x}{x}}$$

$$= \frac{1}{1} = 1$$ Ans.

(vii) $\displaystyle \lim_{x\to 0} \frac{1 - \cos 2x}{x^2}$ $\left(\frac{0}{0}\right)$ **form**

$$\lim_{x\to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$ ($\because$ $\cos 2x = 1 - 2\sin^2 x \Rightarrow 2\sin^2 x = 1 - \cos 2x$)

$$= \lim_{x \to 0} 2\left(\frac{\sin x}{x}\right)^2$$

$$= 2(1)^2 = 2 \quad \text{Ans.}$$

(viii) $\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x}$ $\left(\frac{0}{0}\right)$ **form**

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{1 - \cos^2 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{(1 + \cos x)(1 - \cos x)}$$

$$= \lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= \frac{1}{1 + 1} = \frac{1}{2} \quad \text{Ans.}$$

(ix) $\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta}$ $\left(\frac{0}{0}\right)$ **form**

$$\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \sin \theta$$

$$= 1 \times 0 = 0 \quad \text{Ans.}$$

(x) $\lim_{x \to 0} \frac{\sec x - \cos x}{x}$ $\left(\frac{0}{0}\right)$ **form** ***(G.B 2007)***

$$\lim_{x \to 0} \frac{\sec x - \cos x}{x} = \lim_{x \to 0} \frac{\frac{1}{\cos x} - \cos x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1 - \cos^2 x}{\cos x}}{x}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \tan x = 1 \times 0 = 0 \quad \text{Ans.}$$

(xi) $\lim_{\theta \to 0} \frac{1 - \cos p\theta}{1 - \cos q\theta}$ $\left(\frac{0}{0}\right)$ **form** ***(G.B 2006)***

We know that:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos\theta = 1 - 2\sin^2\frac{0}{2}$$

$$\cos p\theta = 1 - 2\sin^2\frac{p\theta}{2}$$

$$2\sin^2\frac{p\theta}{2} = 1 - \cos p\theta \quad \text{and} \quad 2\sin^2\frac{q\theta}{2} = 1 - \cos q\theta$$

$$\lim_{\theta \to 0} \frac{1 - \cos p\theta}{1 - \cos q\theta} = \lim_{\theta \to 0} \frac{2\sin^2\frac{p\theta}{2}}{2\sin^2\frac{q\theta}{2}}$$

$$= \lim_{\theta \to 0} \frac{\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}} \times \frac{p\theta}{2}\right)^2}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}} \times \frac{q\theta}{2}\right)^2}$$

$$= \lim_{\theta \to 0} \frac{\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2 . \frac{p^2\theta^2}{4}}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2 . \frac{q^2\theta^2}{4}} = \lim_{\theta \to 0} \frac{\left(\frac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2 . p^2}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2 . q^2}$$

$$= \frac{(1)^2 . p^2}{(1)^2 . q^2} = \frac{p^2}{q^2} \quad \text{Ans.}$$

(xii) $\lim_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$ $\left(\frac{0}{0}\right)$ **form** ***(L.B 2005)***

$$\lim_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\cos\theta} - \sin\theta}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta\left(\frac{1}{\cos\theta} - 1\right)}{\sin^3\theta}$$

$$= \lim_{\theta \to 0} \frac{\frac{1 - \cos\theta}{\cos\theta}}{\sin^2\theta}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\cos\theta\,(1 - \cos^2\theta)}$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\cos\theta(1 + \cos\theta)(1 - \cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{1}{\cos\theta(1 + \cos\theta)} = \frac{1}{1(1 + 1)} = \frac{1}{2} \quad \text{Ans.}$$

Q.4 Express each limit in terms of e:

(i) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{2n}$ (ii) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{n/2}$

(iii) $\lim_{n \to +\infty} \left(1 - \frac{1}{n}\right)^{n}$ (iv) $\lim_{n \to +\infty} \left(1 + \frac{1}{3n}\right)^{n}$

(v) $\lim_{n \to +\infty} \left(1 + \frac{4}{n}\right)^{n}$ (vi) $\lim_{x \to 0} (1 + 3x)^{2/x}$

(vii) $\lim_{x \to 0} (1 + 2x^2)^{1/x^2}$ (viii) $\lim_{h \to 0} (1 - 2h)^{1/h}$

(ix) $\lim_{x \to \infty} \left(\frac{x}{1 + x}\right)^{x}$ **(L.B 2003,04)** (x) $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$, $x < 0$

(xi) $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$, $x > 0$ ***(L.B 2005)***

Solution:

(i) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{2n} = \lim_{n \to -\infty} \left[\left(1 + \frac{1}{n}\right)^{n}\right]^2$

$= e^2$ Ans.

(ii) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{n/2} = \lim_{n \to -\infty} \left[\left(1 + \frac{1}{n}\right)^{n}\right]^{1/2}$

$= c^{1/2}$ Ans.

(iii) $\lim\limits_{n \to +\infty} \left(1 - \frac{1}{n}\right)^n = \lim\limits_{n \to +\infty} \left[\left(1 + \left(\frac{1}{-n}\right)\right)^{-n}\right]^{-1}$

$= e^{-1}$ Ans.

(iv) $\lim\limits_{n \to +\infty} \left(1 + \frac{1}{3n}\right)^n = \lim\limits_{n \to +\infty} \left[\left(1 + \frac{1}{3n}\right)^{3n}\right]^{1/3}$

$= e^{1/3}$ Ans.

(v) $\lim\limits_{n \to +\infty} \left(1 + \frac{4}{n}\right)^n = \lim\limits_{n \to +\infty} \left[\left(1 + \frac{1}{n/4}\right)^{n/4}\right]^4$

$= e^4$ Ans.

(vi) $\lim\limits_{n \to 0} (1 + 3x)^{2/x} = \lim\limits_{n \to 0} [(1 + 3x)^{1/3x}]^{2\times 3}$

$= e^6$ Ans.

(vii) $\lim\limits_{x \to 0} (1 + 2x^2)^{1/x^2} = \lim\limits_{x \to 0} [(1 + 2x^2)^{1/2x^2}]^2$

$= e^2$ Ans.

(viii) $\lim\limits_{h \to 0} (1 - 2h)^{1/h} = \lim\limits_{h \to 0} [(1 + (-2h)^{-1/2h}]^{-2}$

$= e^{-2}$ Ans.

(ix) $\lim\limits_{x \to \infty} \left(\frac{x}{1 + x}\right)^x = \lim\limits_{x \to \infty} \left(\frac{1 + x}{x}\right)^{-x}$ ***(G.B 2006) (L.B 2007)***

$= \lim\limits_{x \to \infty} \left(\frac{1}{x} + \frac{x}{x}\right)^{-x}$

$= \lim\limits_{x \to \infty} \left[\left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1}$ Ans.

(x) $\lim\limits_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$, $x < 0$ ***(G.B 2005)***

Put, $x = -t$, where $t > 0$

As, $x \to 0$, $t \to 0$

$\lim\limits_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim\limits_{t \to 0} \frac{e^{1/-t} - 1}{e^{1/-t} + 1}$

$= \frac{e^{-1/0} - 1}{e^{-1/0} + 1} = \frac{e^{-\infty} - 1}{e^{-\infty} + 1}$

$= \frac{0 - 1}{0 + 1} = \frac{-1}{1} = -1$ Ans.

(xi) $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$, $x > 0$ $\left(\dfrac{\infty}{\infty}\right)$

$$\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \to 0} \frac{e^{1/x}\left(1 - \frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1 + \frac{1}{e^{1/x}}\right)}$$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}}$$

$$= \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}}$$

$$= \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1 \qquad \text{Ans.}$$

Continuous Function

A function f is said to be continuous at a number "c" if and only if the following three conditions are satisfied.

(i) $f(c)$ is defined.

(ii) $\lim\limits_{x \to c} f(x)$ exists.

(iii) $\lim\limits_{x \to c} f(x) = f(c)$

EXERCISE 1.4

Q.1 Determine the left hand limit and right hand limit and then find limits of the following functions at x = c.

(i) $f(x) = 2x^2 + x - 5$, $c = 1$

(ii) $f(x) = \dfrac{x^2 - 9}{x - 3}$, $c = -3$

(iii) $f(x) = |x - 5|$, $c = 5$

Solution:

(i) $f(x) = 2x^2 + x - 5$, $c = 1$

Left hand limit